Notebook Assignments

These six assignments are a notebook using techniques from class in the single concrete context of graph theory. This is supplemental to your usual assignments, and is designed for enrichment. You will receive credit for turning in an attempt for each assignment, but at the end of the quad I will also grade for the content. So, if you have a bad or busy week, you can make up the material, but if you let it all wait to the end, you will not get full credit even if it's perfect.

Here is how it works. You will turn the (whole) notebook in for each assignment, on the dates listed below. You will have tried (but perhaps not conquered) all the questions and explorations in that assignment. The way I indicate them is like this: any time you see a direction in **bold type**, you should do it! You may be asked to **verify** that a theorem holds in an example, to **make up** an example, or to (attempt to) **prove** a theorem. Do not fear - the feel will be very exploratory and, hopefully, fun!

Then, at the end of the quad, you will hand in the whole notebook; at that point, you will have responded to my comments, and hopefully made further progress by that time. The better you write (in English) and the more you have discovered in each assignment, the better your grade. That means you should *write in complete sentences* whenever possible! This is especially true on proofs, or when explaining what you have tried.

I suggest you actually buy a small notebook, maybe one of those half-size or quarter-size spiral-bound ones, in which you can doodle and try different things for these assignments. Whatever form you keep it in, it should be legible and I should be able to find the different parts of each assignment with ease.

Tentative due dates for the first attempts at the assignments are as follows:

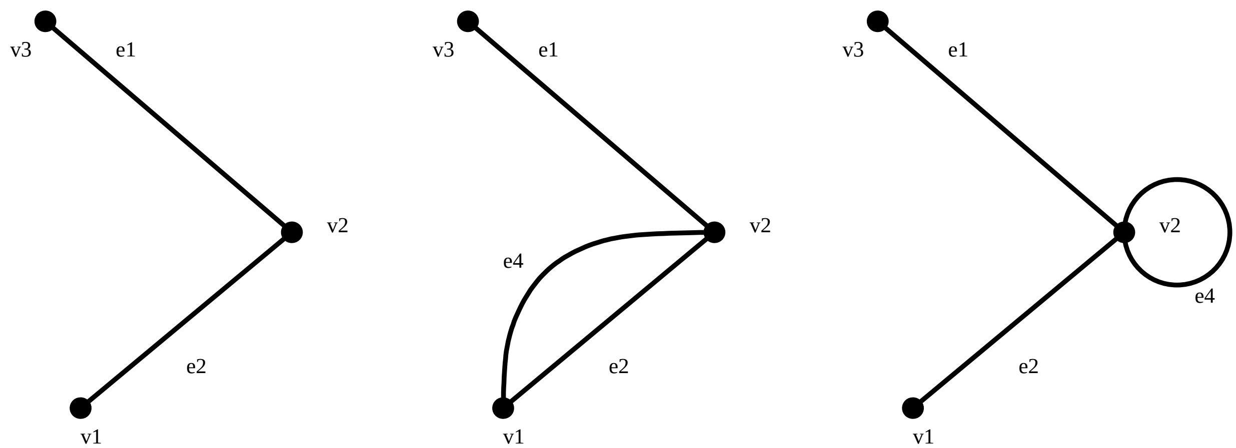
First Assignment	Week 2
Second Assignment	Week 3
Third Assignment	Week 4
Fourth Assignment	Week 6
Fifth Assignment	Week 7
Last Assignment	At Final

NOTE: Of course all of this material is found in many, many books, including some in our library, as well as all over the Internet. But, I assume you are in this class because you want to *do* some math, not just copy it from someplace else. I am trying to keep these short and to the point, so I would appreciate if you really do try them on your own. In the interest of my own intellectual honesty, I will note that my main reference for this material is Gary Chartrand's *Introductory Graph Theory* in its 1985 Dover reprint. Robin Wilson's *Introduction to Graph Theory* and most discrete mathematics textbooks (at any level), along with many (but not all) 'transitions' class books, are also good references. I do not claim originality for any of it except some wording and the pictures, which I created using a Fink build of `xfig`.

1 First Assignment: Graph Theory Intro

Naively speaking, a *graph* is a collection of *vertices* which may or may not be connected by *edges*, which we represent by drawing a picture with dots connected by line segments. We'll have to wait for a more rigorous definition.

So for now, let's take a finite set V of vertices, and a finite set E of edges such that each edge e has two *different* vertices v_1 and v_2 associated with it, and let's call such a thing a graph[1]. Here are some examples of the pictures representing graphs. One of these is not a graph - **which one? Why? Write down** the vertex and edge sets for the two actual graphs, along with the vertices joined by each edge.



G1 G2 G3

We need some terminology! We say an edge e *joins* the vertices associated with it, in which case those vertices are *adjacent*. The number of vertices in V is often called the *order* of the graph, while the number of edges is called the *size* of the graph. A very important concept is the number of distinct edges joined to a specific vertex v; we call this number the *degree* of the vertex v.

The rest of this assignment is just more playing with graphs. Pictures are fine; if it took you a little time to construct something, you should explain how you figured it out.

1. **Draw graphs** - one of order five and size ten, another of order four and size seven. Can you give examples of graphs of those sizes and orders where each edge joins a different pair of vertices (i.e. no 'double' edges)?

2. For each of $n = 2, 3, 4, 5$, **draw** a graph of order n such that at least one vertex is adjacent to all other vertices in the graph. **What do you think** is the *minimum* size for a graph with this property?

3. **Give the degrees** of the vertices for all of the graphs we have looked at up to now. **Mention** any patterns you see (and try more if you do!).

4. **Draw a graph** where V is the set of campus buildings you visit today, with an edge in E for each time you walk from one destination to the next. (What does the 'no loops' condition imply for your day of walking?)

[1]Some of you may know that sometimes you do allow the vertices to be the same, and sometimes you only allow one edge for a given pair of vertices. But in this notebook, this is the definition.

2 Second Assignment: Degrees

Now you have a little experience with graphs, and you also have a little experience with direct proofs. In this assignment you will learn a little more about graphs while trying out some of your mettle. You will try to prove one theorem and *formulate* another one.

Theorem: For any graph G, the sum of the degrees of the vertices is twice the number of edges of G. With n vertices and q edges, this can be written

$$\sum_{i=1}^{n} \text{degree } v_i = 2q\,.$$

Verify the theorem for six of the graphs from the first assignment. Next, **make up** two more larger graphs (order and size at least 6) and verify it. Finally, **try drawing** a graph of size 3 with all vertices having degree 3 (recall the definition of degree if you need), and **explain** in your own words what goes wrong. **Try drawing** a graph with all vertices degree 2, but more edges than vertices, and **explain** in your own words what goes wrong. Now **do your best to prove the theorem** - the easiest is a direct proof. If you are having trouble, first try to prove that the sum of the degrees of the vertices is *even*; this should be easier. You may also wish to try further examples before attempting a proof.

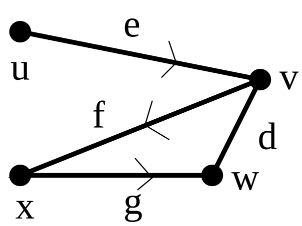
For our exploration, we call a vertex of even degree an *even vertex* and a vertex of odd degree an *odd vertex*. For the six graphs you verified the above theorem for, **list** how many odd vertices and how many even vertices there are for each. Next, **draw (or explain why you can't draw)** graphs with no odd vertices, one odd vertex, two odd vertices, no even vertices, one even vertex, and two even vertices.

Write down any pattern you see at this point. **Identify** what examples you would want to try next to see if the pattern continues. **Do** those examples.

Now, **write down** a guess at what a theorem might be for how many odd or even vertices are possible in a graph. Finally, using your theorem try, **tell me** if there is a graph with vertex degrees 2, 3, 3, 4, 4, and 5. Does your theorem yield a prediction about the existence of graphs with vertex degrees 3, 3, and 3, or 4, 4, and 4?

(As a bonus, can you prove this theorem using the first theorem?)

3 Third Assignment: Circuits

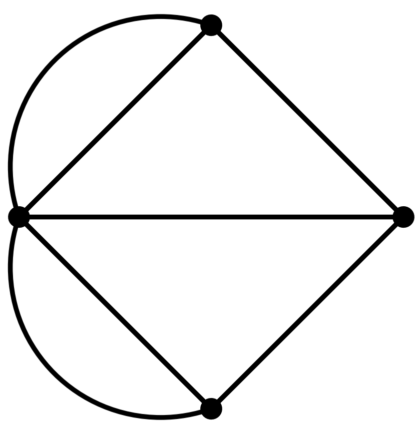


A *walk* of length n is a list of $n+1$ vertices, alternating with n edges, such that two consecutive vertices in the walk are joined by the edge between them in the list. In the graph pictured, $uevfxgw$ is a walk of length three from u to w. In a walk, if no edges are repeated (we say they are all distinct), we call the walk a *trail*; if no vertices are repeated, we call it a *path*. This walk is actually a path.

Pick one of the larger graphs you have already used in a previous assignment. **Give an example of each of the following** from that graph: a walk, a walk which is not a trail, a walk which is not a path, a trail which is not a path, and an actual path. Make sure to **give** the lengths and lists of vertices and edges!

We say that a graph is *connected* if there is a path between *any* two vertices in the graph. That means that, given two vertices u and v, there is a path P between them; of course, you may need a different path to connect two other vertices w and x. **List** several connected graphs you have made so far in the notebook; you do not need to list every path between two vertices! Can you draw a few *disconnected* graphs (that is, not connected)? If so, **draw them and list** two vertices which have no path between them.

Choose three of your previous connected graphs, and **try to find** trails which hit every vertex, or every edge! Try to **explain** why you can't if it isn't possible. A trail is a *circuit* if it begins and ends at the same place. On the same graphs, find some circuits.



Koenigsberg Bridges Graph E

This is a famous graph involving circuits. Look its story up online and **say** what you think about the story! Here, the goal was to find a circuit (remember, no repeated edges) containing *all* the edges and vertices of E. If a graph has this property, we call it *Eulerian*. Is E Eulerian? If so, **give** the circuit. If not, **try to find** a *trail* containing all the edges

Now, we need to practice some proofs too! Here are two facts.

1. An Eulerian graph must be connected.

2. In an Eulerian graph, every vertex has even degree.

The first one you should have little trouble proving - **draw** some examples to see how, then **prove** it. The second one is straightforward, but may take you some effort and drawing some actual circuits before you see what to **do**.

4 Fourth Assignment: Trees

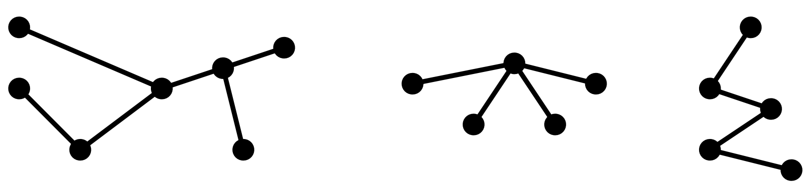
This assignment's goal is a little practice with proof by contradiction, as well as to introduce you to a major type of graph which shows up a lot in both pure mathematics and applied areas such as operations research and computer science. To start, let's review some terminology. A walk in which no edges are repeated is called a *trail*, and a trail which begins and ends at the same place is called a *circuit*.

Here's a new concept: a circuit which repeats only the first (and hence last) vertex is called a *cycle*. **Draw** three cycles (of length at least 3) in a graph of your choice, and also find at least one circuit which is *not* a cycle. **Tell me** how many edges used in the cycle touch any given vertex in the cycle.

A *Hamiltonian cycle* is a cycle which uses each vertex of the graph. Does E from the third assignment have a Hamiltonian cycle? **Find out about and say what you think about** the stories of the "Around the World" and "Traveling Salesman" problems in relation to Hamiltonian cycles.

Now it's time for contradiction. **Draw** the graph represented with vertex set $V = \{u, v, w, x, y\}$ and edge set $E = \{(uw), (vw), (wx), (xu), (xy), (yw)\}$. **Prove** this graph does *not* have a Hamiltonian cycle (and not by drawing all possible cycles). First, state what you would assume in order to start a proof by contradiction. Next, we'll have to think of something to contradict. Here, try to contradict the "Tell me" you discovered two paragraphs ago. In order to do that, it is likely you will need to use the defining property of a Hamiltonian cycle (as opposed to any old cycle). Try your best for now; this may be challenging with little context.

Our other concept for this assignment is that of trees. A *tree* is a connected graph with no cycles. **Draw** trees (other than the examples in the picture below) of order five, six, and seven.



Some examples of trees

List the size and order of all trees you have, both the ones you drew and the ones in the picture. Is there a relationship between the size and the order of a tree? You don't have to prove anything right now. Also, **tell me** why you think such graphs are called trees!

You'll now **prove the following** by contradiction. I have hints for the proof below. **Theorem**: There is *exactly* one path between any pair of vertices u and v in a tree G.

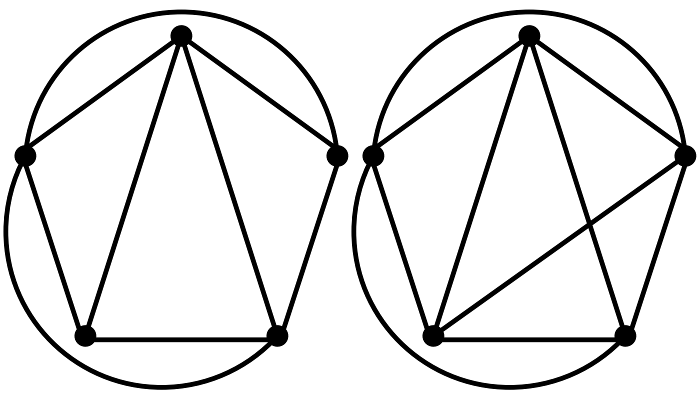
Hints: First, given any two vertices in a tree, why is there *at least* one path between them? Remember, there are two conditions for a graph to be a tree, so see if one of them helps. Next, what should we assume is true (but isn't)? It's something about the number of paths between two vertices. Now it gets tougher, as this assumption must contradict something else we know is true. Since we already used one of the conditions for a graph to be a tree, you may wish to see if it contradicts the *other* condition for a graph to be a tree. On the other hand, you may want to draw some graphs which *do* satisfy the assumption you made (about the number of paths between two vertices), and see why those graphs can't be trees.

5 Fifth Assignment: Planes and Colors

In this assignment we will encounter more advanced concepts, and our technique of proof will need to be induction[2]; usually, it will be induction on the order of the graph.

It is an interesting theorem (to mathematicians, not map-makers) that any normal map on the Earth can be colored with four (or fewer) colors so that no two neighboring regions have the same color. It turns out that the *proof* of this theorem (currently) depends heavily on programming a computer to check thousands and thousands of tedious cases; thus, some people still don't trust the proof, thirty years later.

What does this have to do with graphs? We say an assignment of colors to the vertices of a graph is a *coloring* if adjacent vertices (those connected by an edge) always have different colors assigned. You can think of a vertex as a region and an edge as a boundary between two regions. Pick four graphs, and **color** them (you can just use letters as labels, unless you want to get artistic, which is fine). **Discover and prove** what the *largest* number of colors you can use is (for a given graph).



On the other hand, we usually try to find *minimal* colorings; that is, we try to find a legitimate coloring with the fewest possible colors. The graph on the right needs five colors, but the one on the left only needs four. **Find** the colorings!

Trees are a little special. **Color** the trees from the previous assignment. How colors did you use - was it the fewest possible? Trying a few more if you need to, **formulate** a possible theorem (or *conjecture*) about minimal tree colorings. Now **prove it**! You will probably want to use induction on the order of the graph.

There is another way this relates to graphs. Every graph can be drawn in space without crossing itself, but only some graphs can be drawn in *the plane* without intersecting at points away from vertices. If it can be, we call it *planar*. The graph on the right is not planar, but the one on the left is, although the only difference is an extra edge on the right.

The map theorem above is equivalent to saying that a planar graph can always be colored with four colors (or fewer). Since the one on the right needs five colors, this immediately proves it is not planar! By the way, there is another small graph, of order 6 and size 9, which is nonplanar, like the five vertex, 10 edge graph on the right; **try to find it** by trial and error. On the other hand, not only do they require a small number of colors, trees are actually planar as well, and you can prove it.

First try some examples, then **prove** by induction on the order (number of vertices). You should start with trees of 1 and 2 vertices - are they planar? Then try to prove it for a tree of k vertices; what is your induction assumption? The key concept will be how to find 'room' for another edge on the tree in the plane, and the key difficulty will actually be seeing why this needs to be proved! It does require a proof.

[2]For more induction practice, try to prove your conjecture on the relationship between size and order of a tree from the last assignment.

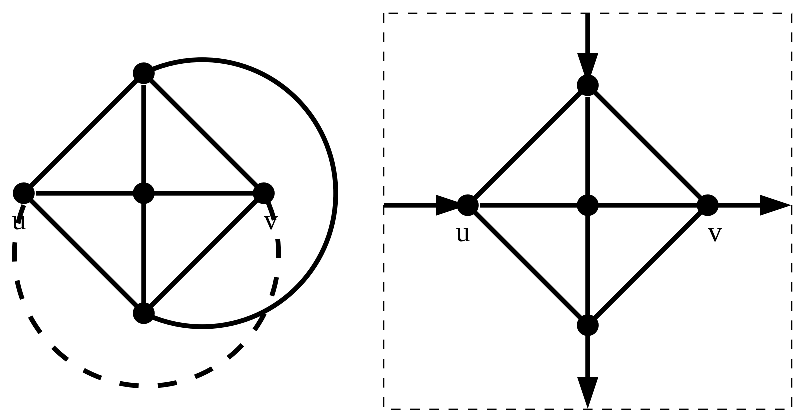
6 Last Assignment: Relations

In ending the notebook we look backward, to finally properly define a graph; we also look forward, to see new directions in where graphs can live. Both will be done using the tool of relations, which is one of the most abstract, but most powerful, tools we have in higher mathematics.

Let's restrict our attention to graphs with no 'double edges' (2-cycles, if you will), and say a relation R on a set X is *irreflexive* if xRx is not true for any element $x \in X$. Then we can define a graph to be a finite nonempty set V (vertices) together with a irreflexive, symmetric relation R on the set V. The edge set E of the graph corresponds to the symmetric pairs under R. For instance, the very first example in the first assignment could be $V = \{v_1, v_2, v_3\}$ with $R = \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2)\}$. Then E has one element from the first pair with v_1 and v_2, and one from the second pair with v_2 and v_3.

Analyze at least three more graphs we've done so far in this way. What happens if you allow a relation which is not irreflexive (or one that is actually reflexive)? **Write** such a relation and **try** to draw the graph which would correspond to this. **Brainstorm** what concept you think a never symmetric relation corresponds to.

It turns out that we can take this a step further. If we have two graphs G and G' with vertex sets V and V', we say G is *isomorphic* to G' if there is a bijection from V to V' such that two vertices u and v are adjacent in G if and only if their images are adjacent in G'.



On the left, we have attempted to draw a graph in the plane, in vain. On the right, we have drawn the same graph successfully in a square where you can continue from one side to the opposite one, like in many video games (including the original Pacman). Now we can draw the graph without crossings! **Show** that these graphs are isomorphic.

Now **draw** two graphs that don't 'look' the same, but are isomorphic (pointing out in what way they 'look' different, and how you know they are isomorphic).

On a more technical note, if we let *isomorphism* be a relation on the set of all graphs, it turns out *that* is an equivalence relation! **Prove** this relation satisfies the axioms for an equivalence relation! It is a little tedious, but very well worth while.

The graph in the figure needs five colors to be colored (try it!). That's okay, since this graph isn't planar[3]; all this obversation says is that the four-color theorem (see previous assignment) isn't true on the Pacman square[4]. **Try** to find a graph living on the Pacman square which needs *six* colors to be colored, with at least three tries.

[3] Can the 'other small graph' in the previous assignment live on the Pacman square?

[4] Extra Credit: Explain in detail how to get the Pacman square itself via an equivalence relation.