9.5 Predator - Prey Equations

Two species:

prey $x(t)$

grows $\frac{dx}{dt} = ax$ $a > 0$ in absence of predators

Predators $y(t)$

shrinks $\frac{dy}{dt} = -cy$ $c > 0$ in absence of prey

Assume # of contacts between predator and prey is proportional to product of their populations - and each contact tends to increase the predator population and decrease the prey population.

$$\frac{dx}{dt} = ax - \alpha xy = x(a - \alpha y)$$
$$\frac{dy}{dt} = -cy + \gamma xy = y(-c + \gamma x)$$

Lotka - Volterra Predator - Prey equations

$F(x,y) = x(a - \alpha y)$ $\qquad$ $G(x,y) = y(-c + \gamma x)$

$F_x = a - \alpha y$ $\qquad$ $G_x = \gamma y$

$F_y = -\alpha x$ $\qquad$ $G_y = -c + \gamma x$

Let (x_0, y_0) be a critical point and $u = x - x_0$, $v = y - y_0$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} a - \alpha y_0 & -\alpha x_0 \\ \gamma y_0 & -c + \gamma x_0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Find eigenvalues and eigen vectors of

#2 $x' = x(1-0.5y)$
$y' = y(-0.25+0.5x)$ $\qquad F(x,y) = x - 0.5xy$, $G(x,y) = -0.25y + 0.5xy$

$x'=0 = x(1-0.5y) \rightarrow x=0$ or $y=2$
$y'=0 = y(-0.25+0.5x) \rightarrow y=0$ or $\frac{1}{4} = \frac{1}{2}x \rightarrow x = \frac{1}{2}$

If $x=0$ then $y=0$. If $x=\frac{1}{2}$ then $y=2$
So

Critical points are $(0,0)$ and $(\frac{1}{2}, 2)$

Corresponding linear system is found using Jacobian

$F_x(x,y) = 1-0.5y \qquad F_y(x,y) = -0.5x$
$G_x(x,y) = 0.5y \qquad G_y(x,y) = -0.25+0.5x$

at $(0,0)$ system is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & -0.25 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$ so $r = 1, -0.25$

$\xi^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \xi^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Saddle Point

at $(\frac{1}{2}, 2)$ system is

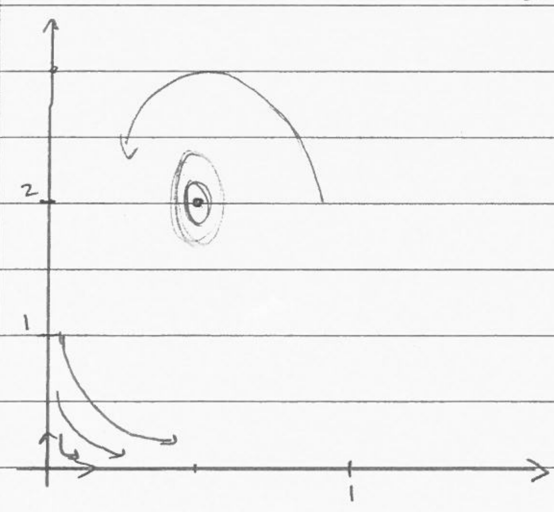
$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & -0.25 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$(-r)(-r) + 0.25 = 0 \rightarrow r^2 = -0.25 \qquad r = \frac{i}{2}, -\frac{i}{2}$

$r = \frac{1}{2}i \rightarrow \begin{bmatrix} -i/2 & -0.25 \\ 1 & -i/2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{\xi}^{(1)} = \begin{bmatrix} i \\ 2 \end{bmatrix} \quad \vec{\xi}^{(2)} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

$\xi_1 = \frac{i}{2}\xi_2 \rightarrow \vec{\xi} = \begin{bmatrix} i \\ 2 \end{bmatrix}$

Center



#5 $\frac{dx}{dt} = x(-1 + 2.5x - 0.3y - x^2)$

$\frac{dy}{dt} = y(-1.5 + x)$

(a) find critical points $\quad 0 = x(-1 + 2.5x - 0.3y - x^2) \quad (*)$

$0 = y(-1.5 + x) \quad (**)$

1. $\boxed{(0,0)}$

2. $y \neq 0 \Rightarrow x = 1.5$ From $(**)$

From $(*)$ we see $-1 + 2.5(1.5) - 0.3y - (1.5)^2 = 0 \quad y = 5/3 = 1.\overline{6}$

$\boxed{(1.5, 1.\overline{6})}$

3. $y = 0$ then $0 = x(-1 + 2.5x - x^2)$

if $x \neq 0$ then $x^2 - 2.5x + 1 = 0 \quad x = \frac{2.5 \pm \sqrt{(2.5)^2 - 4}}{2}$

$= \frac{2.5 \pm 1.5}{2} = \frac{1}{2}, 2$

$\boxed{(\tfrac{1}{2}, 0) \text{ and } (2,0)}$

$\boxed{\text{4 critical points: } (0,0),\ (\tfrac{1}{2},0),\ (2,0),\ (\tfrac{3}{2}, \tfrac{5}{3})}$

(b) $(0,0)$: $\quad \vec{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1.5 \end{bmatrix} \vec{x} \quad \rightarrow \quad r_1 = -1, \quad r_2 = -1.5$

$\vec{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{\xi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

critical point in a node, asymptotically stable

$F_x = -1 + 5x - 0.3y - 3x^2 \qquad F_y = -0.3x$

$G_x = y \qquad G_y = -1.5 + x$

$(\frac{1}{2}, 0)$: $\vec{u}' = \begin{bmatrix} 3/4 & -\frac{3}{20} \\ 0 & -1 \end{bmatrix} \vec{u} \rightarrow r_1 = \frac{3}{4} \quad r_2 = -1$

$\vec{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{\xi}_2 \simeq \begin{bmatrix} 3 \\ 35 \end{bmatrix}$

critical point is saddle point, unstable

$(2, 0)$: $\vec{u}' = \begin{bmatrix} -3 & -\frac{3}{5} \\ 0 & \frac{1}{2} \end{bmatrix} \vec{u} \rightarrow r_1 = \frac{1}{2} \quad r_2 = -3$

$\vec{\xi}_1 = \begin{bmatrix} -6 \\ 35 \end{bmatrix} \quad \vec{\xi}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

critical point is saddle point, unstable

$(\frac{3}{2}, \frac{5}{3})$: $\vec{u}' = \begin{bmatrix} -3/4 & -9/20 \\ 5/3 & 0 \end{bmatrix} \vec{u} \rightarrow r_1 = -\frac{3}{8} - \frac{\sqrt{39}}{8} i, \; r_2 = -\frac{3}{8} + \frac{\sqrt{39}}{8} i$

$\vec{\xi}_1 = \begin{bmatrix} -0.225 - 0.46838 i \\ 1 \end{bmatrix} \quad \vec{\xi}_2 = \begin{bmatrix} -0.225 + 0.46838 i \\ 1 \end{bmatrix}$

critical point is spiral point, asymptotically stable.

$$\vec{u}^{(1)} = e^{-\frac{3}{8}t} \left(\begin{bmatrix} -0.225 \\ 1 \end{bmatrix} \cos \frac{\sqrt{39}}{8} t - \begin{bmatrix} 0.46838 \\ 0 \end{bmatrix} \sin \frac{\sqrt{39}}{8} t \right)$$

$$\vec{u}^{(2)} = e^{-\frac{3}{8}t} \left(\begin{bmatrix} -0.225 \\ 1 \end{bmatrix} \sin \frac{\sqrt{39}}{8} t + \begin{bmatrix} 0.46838 \\ 0 \end{bmatrix} \cos \frac{\sqrt{39}}{8} t \right)$$

$\vec{u}^{(1)}(0) = \begin{bmatrix} -0.225 \\ 1 \end{bmatrix}$ As t increases

component in $-\begin{bmatrix} 0.46838 \\ 0 \end{bmatrix}$ direction increases,

giving counterclockwise rotation.

9.5

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