9.4 Competing Species

Recall the population growth models we've already seen:

① $\frac{dy}{dt} = ry$ r is growth rate   exponential growth

② $\frac{dy}{dt} = (r - ay)y$ r/a is saturation level   logistic equation

The second equation is more accurate in that it accounts for slowing in growth and a limiting maximum population due to limited resources.

Now consider two species with populations given by $x(t)$ and $y(t)$. If these species do not compete for resources then they can be modeled with

$$x' = x(\varepsilon_1 - \sigma_1 x)$$
$$y' = y(\varepsilon_2 - \sigma_2 y)$$

If, however, they compete, then not only should the "growth factor" for x be reduced by $\sigma_1 x$, it should also be reduced by $\alpha_1 y$. A similar adjustment should be made for the y equation

$$x' = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$y' = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x)$$

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1. $\dot{x} = x(1.5 - x - 0.5y)$

$\dot{y} = y(2 - y - 0.75x)$

a) find critical points

$0 = x(1.5 - x - 0.5y) \rightarrow x = 0$ or $1.5 = x + 0.5y$

$0 = y(2 - y - 0.75x) \rightarrow y = 0$ or $2 = y + 0.75x$

if $x = 0$ and $y \neq 0 \rightarrow 2 = y$

if $y = 0$ and $x \neq 0 \rightarrow 1.5 = x$

$2x + y = 3$
$3x + 4y = 8$ $\rightarrow$ $x = 0.8$
$y = 1.4$

So the four critical points are $(0,0)$, $(0,2)$, $(1.5,0)$, $(0.8, 1.4)$

b) Note Eq 11 from 9.3

$$\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}' = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \cancelto{0}{\begin{bmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{bmatrix}}$$

We can find the linear system associated with each critical point (x_0, y_0) of an almost linear system.

$x' = F(x,y) = x(1.5 - x - 0.5y)$

$y' = G(x,y) = y(2 - y - 0.75x)$

$F_x = 1.5 - 2x - 0.5y$

$F_y = -0.5x$

$G_x = -0.75y$

$G_y = 2 - 2y - 0.75x$

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- So, for critical point $(0,0)$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow r_1 = 1.5, \quad r_2 = 2$$

$$r_1 = 1.5 \rightarrow A - rI = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} \rightarrow 0\xi_1 + 0.5\xi_2 = 0 \rightarrow \vec{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$r_2 = 2.0 \rightarrow A - rI = \begin{bmatrix} -0.5 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{\xi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- critical point $(0,2)$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0.5 & 0 \\ -1.5 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow r_1 = -2, \quad r_2 = 0.5$$

$$\vec{\xi}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{\xi}_2 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

- Critical point $(1.5, 0)$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow r_1 = -1.5 \qquad r_2 = 0.875$$

$$\vec{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{\xi}_2 = \begin{bmatrix} -6 \\ 19 \end{bmatrix}$$

- Critical point $(0.8, 1.4)$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow r_1 = -0.385857157146$$

$$r_2 = -1.81414284285$$

$$\vec{\xi}_1 = \begin{bmatrix} -0.965850326528 \\ 1 \end{bmatrix}$$

$$\vec{\xi}_2 = \begin{bmatrix} 0.394421755099 \\ 1 \end{bmatrix}$$

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