9.3 Locally Linear Systems

Thm The critical point $\vec{x} = \vec{0}$ of $\vec{x}' = A\vec{x}$ is

1. asymptotically stable if the eigenvalues r_1 and r_2 are real and negative or have a negative real part
2. stable but not asymptotically stable if r_1 and r_2 are pure imaginary.
3. unstable if r_1 and r_2 are real and either is positive or if they have positive real part.

In the case that $r_1 < r_2 < 0$ or $r_1 > r_2 > 0$, the critical point $\vec{x} = \vec{0}$ is called a node (or improper node).

When $r_1 = r_2 = r$ we found that either $\vec{x} = \vec{0}$ was a proper node when there were 2 linearly independent eigenvectors or an improper node if only one eigenvector was found.

When $r_1, r_2 = \lambda \pm i\mu$, $\lambda \neq 0$, $\vec{x} = \vec{0}$ is a spiral point since the trajectories either spiral in to or out from it.

Finally, if $\lambda = 0$ then $\vec{x} = \vec{0}$ is the center of the periodic trajectory.

9.3

Consider the harmonic oscillator $m\ddot{x} = -kx$. $k>0$

same as x''

$x_1 = x$, $x_2 = x_1'$

so $x_1' = x_2$ $\rightarrow$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_2' = -\frac{k}{m}x_1$

eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$: $\begin{vmatrix} -r & 1 \\ -\frac{k}{m} & -r \end{vmatrix} = (-r)^2 + \frac{k}{m} = 0$

$r = \pm i\sqrt{\frac{k}{m}}$

$\Rightarrow$ pure imaginary.

$\vec{x} = \vec{0}$ is the <u>center</u>.

What would happen if even the slighest amount of damping was present?

$$m\ddot{x} = -c\dot{x} - kx \qquad c>0, \ k>0$$

$x_1 = x$, $x_2 = x_1'$ $\qquad x_1' = x_2$

$x_2' = -\frac{k}{m}x_1 - \frac{c}{m}x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$ has eigenvalues given by $\begin{vmatrix} -r & 1 \\ -\frac{k}{m} & -\frac{c}{m}-r \end{vmatrix} = 0$

$= (-r)(-\frac{c}{m} - r) + \frac{k}{m} = 0$

$= r^2 + \frac{c}{m}r + \frac{k}{m} = 0$

$$r = \frac{-\frac{c}{m} \pm \sqrt{(\frac{c}{m})^2 - 4\frac{k}{m}}}{2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Assuming $\left(\frac{c}{2m}\right)^2 < \frac{k}{m}$ then the eigenvalues are complex with real part $-\frac{c}{2m}$.

In this case, $\vec{x}=\vec{0}$ is a <u>spiral point</u>.

This is an important example - it is a special case where slightly changing the system leads to an asymptotically different solution.

small change $\rightarrow$ big difference.

Another place this can occur is if $r_1 = r_2 = r$ is a repeated real eigenvalue of some matrix A. A slight change in A will either result in $r_1 \neq r_2$ but real (critical point becomes an <u>improper node</u> if it was not already) or $r_1, r_2 = \lambda \pm i\mu$ (critical point becomes a <u>spiral point</u> or even a <u>center</u>)

<u>Break</u>

Consider the system $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$. This can be assumed to be non linear since otherwise it could be rewritten as $\vec{x}' = \bar{A}\vec{x} + \vec{k}$ for some constant vector $\vec{k}$.

Suppose $\vec{x}^{\circ}$ is a critical point of $\vec{x}' = A\vec{x} + g(\vec{x})$ and set $\vec{x} = \vec{u} + \vec{x}^{\circ}$. Then

$$\vec{u}' = A\vec{u} + A\vec{x}^{\circ} + \vec{g}(\vec{u} + \vec{x}^{\circ})$$

Notice that $\vec{u} = \vec{0}$ is a critical point of this system since

$$A\vec{0} + A\vec{x}^{\circ} + g(\vec{x}^{\circ}) = \vec{0} + \vec{0} = \vec{0}.$$

If $\vec{g}(\vec{x})$ becomes "small" as $\vec{x} \to \vec{x}^{\circ}$ then we would expect the behavior of the solution near $\vec{x}^{\circ}$ to resemble the solutions of a linear system.

If $\vec{g}(\vec{x})$ has first partial derivatives at $\vec{x}^{\circ}$ and if

$$\lim_{x \to x^{\circ}} \frac{\|\vec{g}(\vec{x})\|}{\|\vec{x}\|} = 0$$

then we say $\vec{x}' = A\vec{x} + g(\vec{x})$ is a <u>locally linear system</u> in the neighborhood of $\vec{x}^{\circ}$.

Thm 9.3.2 Allows us to draw conclusions and understand the behavior of a locally linear system near $\vec{x}^{\circ}$.

$\Rightarrow$ Assumes $\vec{x} = \vec{u} + \vec{x}^{\circ}$ substitution has been done so critical point is $\vec{u} = \vec{0}$

In general, the system $\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$ will be locally linear near a critical point (x_0, y_0) when F and G both have continuous partial derivatives up to order 2.

Using a Taylor series expansion in 2 variables it is possible to show that when $\vec{x} = \vec{u} + \vec{x}^0$ the system

$$\begin{matrix} x' = F(x,y) & \text{becomes} & u_1' = F_x(x_0,y_0)u_1 + F_y(x_0,y_0)u_2 + \eta_1(x,y) \\ y' = G(x,y) & & u_2' = G_x(x_0,y_0)u_1 + G_y(x_0,y_0)u_2 + \eta_2(x,y) \end{matrix}$$

or

$$\vec{u}' = J(\vec{x}^0)\,\vec{u} + \vec{\eta}(\vec{x})$$

where the Jacobian matrix J evaluated at $\vec{x}^0$ is given by

$$J(\vec{x}_0) = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}$$

The vector $\vec{\eta}(\vec{x})$ has the property that $\lim_{\vec{x} \to \vec{x}^0} \dfrac{\|\vec{\eta}(\vec{x})\|}{\|\vec{x} - \vec{x}^0\|} = 0$ so we can neglect it near the critical point $\vec{x}^0$.

This means the behavior of $\vec{u}' \doteq J(\vec{x}^0)\vec{u}$ near $\vec{u} = 0$ matches the behavior of $\vec{x} = \begin{bmatrix} F(\vec{x}) \\ G(\vec{x}) \end{bmatrix}$ near $\vec{x}^0$.

Ex. $\frac{dx}{dt} = \underbrace{x - y + xy}_{F}$ $\frac{dy}{dt} = \underbrace{3x - 2y - xy}_{G}$

$F_x = 1 + y$ $F_y = -1 + x$ $G_x = 3 - y$ $G_y = -2 - x$

Critical points: ① $0 = x - y + xy$ ② $0 = 3x - 2y - xy$

Clearly $(0,0)$ is a critical pt.

From ① we have $y = x + xy = x(1+y)$

so $x = y/1+y$ if $y \neq -1$

Substituting this into ② we have

$$0 = 3\left(\frac{y}{1+y}\right) - 2y - \left(\frac{y}{1+y}\right)y = \left(3y - 2y(1+y) - y^2\right)\frac{1}{1+y}$$

So

$$0 = 3y - 2y - 2y^2 - y^2 = y - 3y^2 = y(1-3y)$$

Thus $y = 0$ or $y = 1/3$.

If $y = 0$ we have $x = 0$.

If $y = 1/3$, we have $x = \frac{1/3}{1 + 1/3} = \frac{1/3}{4/3} = 1/4$

Critical points are $(0,0)$, $(\frac{1}{4}, \frac{1}{3})$.

At $(0,0)$: $\begin{bmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$

$\vec{x} = \vec{u} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so locally linear system is $\boxed{\vec{u}' = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}\vec{u}}$

Eigenvalues are $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ $\rightarrow$ c.p. is <u>attracting spiral point</u>

A $\left(\frac{1}{4}, \frac{1}{3}\right)$ $\begin{bmatrix} F_x(\frac{1}{4},\frac{1}{3}) & F_y(\frac{1}{4},\frac{1}{3}) \\ G_x(\frac{1}{4},\frac{1}{3}) & G_y(\frac{1}{4},\frac{1}{3}) \end{bmatrix} = \begin{bmatrix} 4/3 & -3/4 \\ 8/3 & -9/4 \end{bmatrix}$

$\vec{X} = \vec{u} + \begin{bmatrix} 1/4 \\ 1/3 \end{bmatrix}$ and locally linear system is

$$\boxed{\vec{u}\,' = \begin{bmatrix} 4/3 & -3/4 \\ 8/3 & -9/4 \end{bmatrix} \vec{u}}$$

Eigenvalues are $\frac{-11 \pm \sqrt{697}}{24}$ One is positive and one is negative so c.p. is a saddle point