7.8 Repeated Eigenvalues

Consider

$$\vec{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \vec{x}$$

Assume $\vec{x} = \vec{\xi} e^{rt}$

$\vec{x}' = \vec{\xi} r e^{rt}$

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \vec{\xi} = r \vec{\xi}$$

$$\det \begin{bmatrix} 2-r & 1 \\ -1 & 4-r \end{bmatrix} = (2-r)(4-r)+1 = 0$$

$$= 8 - 6r + r^2 + 1 = r^2 - 6r + 9 = 0$$

$$= (r-3)^2 = 0$$

So $r=3$ is an eigenvalue of multiplicity 2.

Now find $\vec{\xi}$.

$$\left[\begin{array}{cc|c} 2-3 & 1 & 0 \\ -1 & 4-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $\xi_1 = \xi_2$

$$\vec{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

only found one eigenvector.

$$\vec{x}^{(1)} = \vec{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

but we don't have a second lin. indep. solution.

Suppose we had assumed $\vec{x} = \vec{\xi} t e^{rt} + \vec{\eta} e^{rt}$. Then

$$\vec{x}' = \vec{\xi} e^{rt} + \vec{\xi} t r e^{rt} + \vec{\eta} r e^{rt}$$

so

$$\vec{x}' = A\vec{x} \rightarrow \quad \vec{\xi} tr + \vec{\xi} + \vec{\eta} r = A(\vec{\xi} t + \vec{\eta})$$

Equating coefficients of like powers of t, we obtain

$$A\vec{\xi} = r\vec{\xi} \qquad \text{and} \qquad A\vec{\eta} = r\vec{\eta} + \vec{\xi}$$

$$(A - rI)\vec{\xi} = 0 \qquad \text{and} \qquad (A - rI)\vec{\eta} = \vec{\xi}$$

This finds $r = 3$ mult. 2 and

$$\vec{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as before.

$\vec{\eta}$ is called a generalized eigenvector

Now, solve

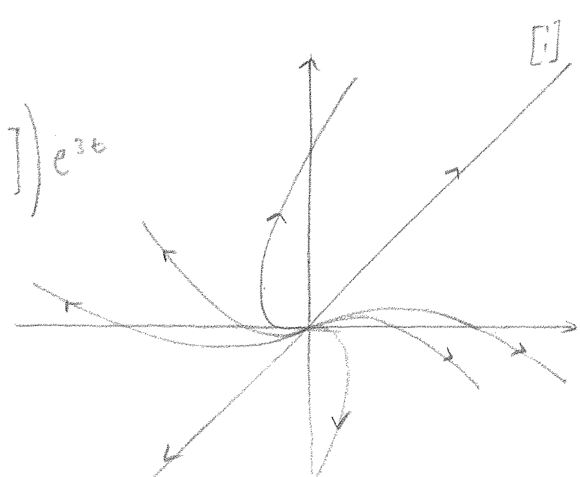
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \quad -\eta_1 + \eta_2 = 1$$

$$\eta_1 = \eta_2 - 1$$

So $$\vec{\eta} = \begin{bmatrix} \eta_2 - 1 \\ \eta_2 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \text{if } k = \eta_2$$

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{3t}}_{\vec{x}^{(2)}} + k\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}}_{\vec{x}^{(1)}}$$

So $$\vec{x} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) e^{3t}$$



In general, to solve $\vec{x}' = A\vec{x}$ we begin as usual, assuming

$$\vec{x} = \vec{\xi} e^{rt}$$

If this leads to an eigenvalue $r = \rho$ of multiplicity m, we next seek eigenvector(s) associated with this eigenvalue.

Case 1: m linearly independent eigenvectors are found... all is well...

Case 2: 1 linearly independent eigenvector is found...

Assume $\vec{x} = \vec{\eta}_{m-1} \frac{t^{(m-1)}}{(m-1)!} e^{\rho t} + \cdots + \vec{\eta}_0 e^{\rho t}$

which will lead to the systems

$$(A - \rho I)\vec{\eta}_{m-1} = \vec{0}$$

$$(A - \rho I)\vec{\eta}_{m-2} = \vec{\eta}_{m-1}$$

$$\vdots \qquad \vdots$$

$$(A - \rho I)\vec{\eta}_0 = \vec{\eta}_1$$

Case 3: Some number (greater than 1 and less than m) of linearly independent eigenvectors are found...
See exercises 17 & 18

Try in class #2, #9