7.7 Fundamental Matrices

Consider the system $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

The eigenvalues and eigenvectors of this system are

$$\lambda_1 = 2 \qquad \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda_2 = -3 \qquad \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

So a fundamental set of solution is

$$\vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} \qquad \vec{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} = \begin{bmatrix} e^{-3t} \\ -4e^{-3t} \end{bmatrix}$$

and the general solution is $\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$

We define the fundamental Matrix $\Psi(t)$ for this problem to be

$$\Psi(t) = \begin{bmatrix} \vec{x}^{(1)}(t) & \vec{x}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix}$$

Since $\vec{x}^{(1)\prime} = A\vec{x}^{(1)}$ and $\vec{x}^{(2)\prime} = A\vec{x}^{(2)}$ we have

$$\boxed{\Psi'(t) = A\Psi(t)}$$

If an initial condition $\vec{x}^0$ is available then $\vec{x}^0 = c_1\vec{x}^{(1)}(0) + c_2\vec{x}^{(2)}(0)$

or

$$\boxed{\vec{x}^0 = \Psi(0)\,\vec{c}} \qquad \text{where } \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

This also lets us write the solution at time t as $\boxed{\vec{x}(t) = \Psi(t)\,\vec{c}}$

We know that $\Psi(t)$ is invertible (why? - its columns are the linearly independent solution vectors). Thus

$$\vec{c} = \Psi(0)^{-1}\vec{x}^0$$

Which means, in turn, that $\boxed{\vec{x}(t) = \Psi(t)\Psi(0)^{-1}\vec{x}^0}$

Ex Suppose $\vec{x}^0 = \begin{bmatrix}1\\6\end{bmatrix}$ for our problem. Then

$$\Psi(0)\vec{c} = \begin{bmatrix}1\\6\end{bmatrix} \rightarrow \begin{bmatrix}e^0 & e^0\\ e^0 & -4e^0\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & -4\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\1 & -4 & 6\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & -5 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & 1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\end{bmatrix} \text{ so } \vec{c} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$\therefore \vec{x}(t) = \begin{bmatrix}e^{2t} & e^{-3t}\\ e^{2t} & -4e^{-3t}\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} = 2\begin{bmatrix}e^{2t}\\e^{2t}\end{bmatrix} - \begin{bmatrix}e^{-3t}\\-4e^{-3t}\end{bmatrix}$$

Note that using $\Psi(t)$ does not save us any work - it only provides a different way to represent the solution.

The matrix $\Psi(t)$ for a given problem is not unique - its column consist of any n linearly independent solution vectors to the linear system.

In particular, $\Phi(t) = \Psi(t)\Psi(0)^{-1}$ is also a fundamental matrix for $\vec{x}' = A\vec{x}$ if $\Psi(t)$ is a fundamental matrix:

$$A\Phi(t) = A\Psi(t)\Psi(0)^{-1} = \Psi'(t)\Psi(0)^{-1} = (\Psi(t)\Psi(0)^{-1})' = \Phi(t)' \Rightarrow \Phi'(t) = A\Phi(t)$$

We can find $\Phi(t)$ by forming $\Psi(t)\Psi^{-1}(0)$, but this requires finding the inverse of an $n\times n$ matrix. This may be worthwhile, however, if the problem

$$\begin{cases} \vec{X}' = A\vec{X} \\ \vec{X}(0) = \vec{X}^0 \end{cases}$$

must be solved for several different initial conditions $\vec{X}^0$. This is because the solution $\vec{X}(t)$ is given by

$$\vec{X} = \Phi(t)\,\vec{X}^0$$

only requiring a matrix-vector multiplication, rather than solving

$$\Psi(0)\,\vec{c} = \vec{X}^0$$

and then computing

$$\vec{X} = \Psi(t)\,\vec{c}\,.$$

Finally note that the columns of $\Phi(t)$ can be computed by solving $\Psi(0)\,\vec{c}^{(i)} = \vec{e}^{(i)}$ for $i=1,\dots,n$ and then forming $\vec{\phi}^{(i)} = \Psi(t)\,\vec{c}^{(i)}$

Ex $\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} c_1^1 \\ c_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \vec{c}^{(1)} = \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}$ $\quad \vec{\phi}^{(1)} = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix}\begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4e^{2t}+e^{-3t} \\ 4e^{2t}-4e^{-3t} \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} c_1^2 \\ c_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \vec{c}^{(2)} = \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix}$ $\quad \vec{\phi}^{(2)} = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix}\begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{2t}-e^{-3t} \\ e^{2t}+4e^{-3t} \end{bmatrix}$

So $\Phi(t) = \frac{1}{5}\begin{bmatrix} 4e^{2t}+e^{-3t} & e^{2t}-e^{-3t} \\ 4e^{2t}-4e^{-3t} & e^{2t}+4e^{-3t} \end{bmatrix}$