7.6 Complex Eigenvalues

Solve $\vec{x}' = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \vec{x}$ assume $\vec{x} = \vec{\xi} e^{rt}$

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \vec{\xi} = r\vec{\xi}$$

Find r: $0 = \det \begin{bmatrix} 5-r & -2 \\ 1 & 3-r \end{bmatrix} = (5-r)(3-r)+2$

$$= 15 - 8r + r^2 + 2$$

$$0 = r^2 - 8r + 17$$

$$r = \frac{8 \pm \sqrt{64-68}}{2} = \frac{8 \pm i\sqrt{4}}{2} = 4 \pm i$$

Find $\vec{\xi}$ corresponding to $r = 4+i$

$$\begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (5-4-i)\xi_1 - 2\xi_2 = 0$$

or

$$(1-i)\xi_1 = 2\xi_2$$

take $\xi_1 = 1+i$

so $2 = 2\xi_2 \rightarrow \xi_2 = 1$

$$\vec{\xi}^{(1)} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Find $\vec{\xi}$ corresponding to $r = 4-i$

$$\begin{bmatrix} 5-(4-i) & -2 \\ 1 & 3-(4-i) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (1+i)\xi_1 - 2\xi_2 = 0$$

$$\rightarrow \xi_1 = 1-i, \quad \xi_2 = 1$$

$$\vec{\xi}^{(2)} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Notice that $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$ are complex conjugates

So $\quad \vec{x}^{(1)} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(4+i)t} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{4t} e^{it}$

$$\vec{x}^{(2)} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(4-i)t} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{4t} e^{-it}$$

Notice $\vec{x}^{(2)} = \overline{\vec{x}^{(1)}}$

$$\vec{x}^{(1)}_{(2)} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t} [\cos t \pm i \sin t]$$

$+$ for $\vec{x}^{(1)}$
$-$ for $\vec{x}^{(2)}$

$$= e^{4t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right) \pm i\, e^{4t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin t \right)$$

these form 2 lin. indep. solutions

$$\vec{x} = e^{4t}\left[c_1 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right) + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t \right) \right]$$

allows for $\pm$

or

$$\vec{x} = c_1 e^{4t} \begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin t + \cos t \\ \sin t \end{bmatrix}$$

In general: $\quad \vec{x}' = A\vec{x} \quad \Rightarrow \quad A\vec{\xi} = r\vec{\xi}$

so $\det(A - rI) = 0$ is the characteristic Equation

If A is real then the char. eq. will have real roots and/or complex roots occuring in conjugate pairs.

In this case, the eigenvectors of A are also conjugates since

$$\overline{A\vec{\xi}} = A\overline{\vec{\xi}} \quad ; \quad \overline{r\vec{\xi}} = \bar{r}\,\overline{\vec{\xi}} \quad \text{so} \quad A\overline{\vec{\xi}} = \bar{r}\,\overline{\vec{\xi}}$$

In this case, $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ will be conjugates as well, so that $\vec{x}^{(1)}+\vec{x}^{(2)}$ is strictly real and $\vec{x}^{(1)}-\vec{x}^{(2)}$ is strictly imaginary.

$$r_1 = \lambda + i\mu$$

$$\vec{\xi}^{(1)} = \vec{a} + i\vec{b} \rightarrow \vec{x}^{(1)} = (\vec{a}+i\vec{b})e^{(\lambda+i\mu)t}$$

$$\text{so} \quad \vec{x}^{(2)} = (\vec{a}-i\vec{b})e^{(\lambda-i\mu)t}$$

$$\vec{x}^{(1)} = (\vec{a}+i\vec{b})e^{\lambda t}[\cos\mu t + i\sin\mu t]$$

$$= e^{\lambda t}[\vec{a}\cos\mu t - \vec{b}\sin\mu t] + i e^{\lambda t}[\vec{a}\sin\mu t + \vec{b}\cos\mu t]$$

$$\vec{u} = \frac{1}{2}[\vec{x}^{(1)}+\vec{x}^{(2)}] = e^{\lambda t}[\vec{a}\cos\mu t - \vec{b}\sin\mu t]$$

$$\vec{v} = \frac{1}{2i}[\vec{x}^{(1)}-\vec{x}^{(2)}] = e^{\lambda t}[\vec{a}\sin\mu t + \vec{b}\cos\mu t]$$

In general, then, the solution to the 2×2 system $\vec{x}' = \overset{\text{real matrix}}{A}\vec{x}$ when the eigenvalues $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$ are complex conjugates, is

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{v} \quad \text{where}$$

$$\left\{\begin{array}{l} \vec{u} = e^{\lambda t}[\vec{a}\cos\mu t - \vec{b}\sin\mu t]; \; \vec{v} = e^{\lambda t}[\vec{a}\sin\mu t + \vec{b}\cos\mu t] \\ \vec{a} = \mathrm{Re}\,\vec{\xi}^{(1)}; \quad \vec{b} = \mathrm{Im}\,\vec{\xi}^{(1)} \\ \text{where } \vec{\xi}^{(1)} \text{ is the complex eigenvector corresponding} \\ \text{to } r_1. \end{array}\right.$$

#2. $\vec{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$

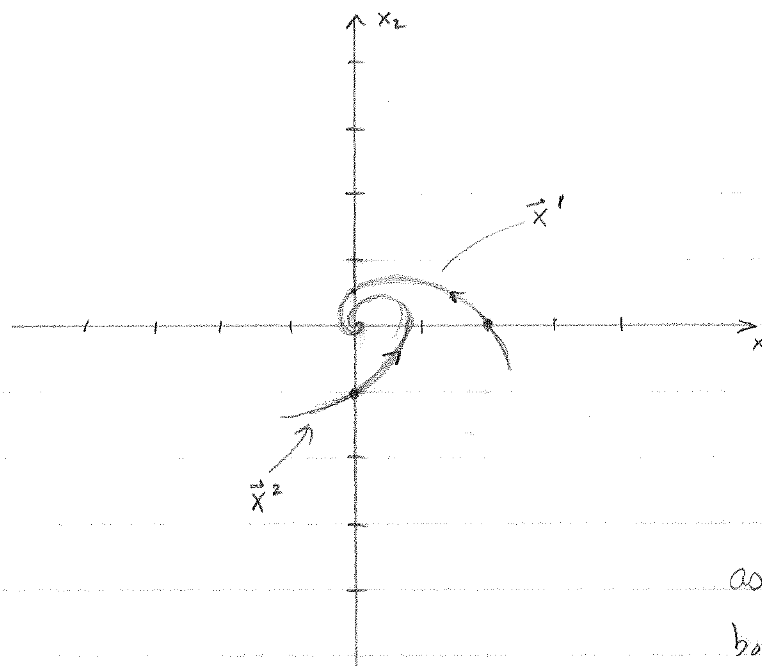
$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$ has eigenvalues $r_1 = -1+2i$, $r_2 = -1-2i$

with eigenvectors $\vec{\xi}^{(1)} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$ $\vec{\xi}^{(2)} = \begin{bmatrix} 2 \\ i \end{bmatrix}$

So $\vec{x}^{(1)} = e^{-t}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin 2t\right)$

$\vec{x}^{(2)} = e^{-t}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos 2t\right)$

$\vec{x} = c_1 e^{-t} \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\sin 2t \\ -\cos 2t \end{bmatrix}$



as $t \to \infty$ $\vec{x}^1$ and $\vec{x}^2$ both go to 0, so $\vec{x} \to \vec{0}$

Notice that, unlike the cases with real eigenvalues that we've seen so far, these solutions "spiral" rather than lining up in particular directions.