7.5 Homogeneous Linear Systems with Constant Coefficients

Consider $\vec{x}' = A\vec{x}$; A is a constant $n \times n$ matrix.

Remember $x' = ax$, assumed a solution $x = e^{rt}$.

Assume a solution $\vec{x} = \vec{\xi} e^{rt}$

$\uparrow$ $n \times 1$ vector (constant)

$$\vec{x}' = \vec{\xi} r e^{rt} = r\vec{x}$$

So $\vec{x}' = A\vec{x} \Rightarrow r\vec{x} = A\vec{x} \Rightarrow r\vec{\xi}e^{rt} = A\vec{\xi}e^{rt} \Rightarrow A\vec{\xi} = r\vec{\xi}$

So r will be an eigenvalue of A and $\vec{\xi}$ is the corresponding eigenvector.

1. Find eigenvalues by solving $\det(A - rI) = 0$
2. Find eigenvectors by solving $(A - rI)\vec{\xi} = \vec{0}$

Ex

$\vec{x}' = \begin{bmatrix} 11 & 4 \\ -12 & -3 \end{bmatrix} \vec{x}$ Assume $\vec{x} = \vec{\xi} e^{rt}$

$\downarrow$

$$0 = \det(A - rI) = \det \begin{bmatrix} 11-r & 4 \\ -12 & -3-r \end{bmatrix} = (11-r)(-3-r) + 48$$

$$0 = -33 - 8r + r^2 + 48 \rightarrow \quad r^2 - 8r + 15 = 0$$

$$(r-3)(r-5) = 0$$

$$r_1 = 3, \quad r_2 = 5$$

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$r_1 = 3$

$$A - 3I = \begin{bmatrix} 8 & 4 \\ -12 & -6 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 8 & 4 & 0 \\ -12 & -6 & 0 \end{array}\right]$$

$$8x_1 + 4x_2 = 0 \rightarrow \quad 2x_1 = -x_2$$

$$x_1 = 1 \;, \quad x_2 = -2$$

$$\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\underline{\underline{\vec{x}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{3t}}}$$

$r_2 = 5$

$$A - 5I = \begin{bmatrix} 6 & 4 \\ -12 & -8 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 6 & 4 & 0 \\ -12 & -8 & 0 \end{array}\right]$$

$$6x_1 + 4x_2 = 0$$

$$3x_1 + 2x_2 = 0 \qquad\qquad x_1 = 2, \quad x_2 = -3$$

$$\vec{\xi}^{(2)} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\underline{\underline{\vec{x}^{(2)} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{5t}}}$$

So, general solution is

$$\boxed{\vec{x} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{5t}}$$

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \begin{vmatrix} e^{3t} & 2e^{5t} \\ -2e^{3t} & -3e^{5t} \end{vmatrix} = -3e^{8t} + 4e^{8t} = e^{8t} \neq 0$$

Thus confirming that we have a fundamental set of solutions

Ex Solve $\vec{x}' = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}$ and draw a sketch of the solution.

Assume $\vec{x} = \vec{\xi} e^{rt} \Rightarrow \vec{x}' = r\vec{\xi} e^{rt}$

$$r\vec{\xi}e^{rt} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \vec{\xi} e^{rt} \Rightarrow A\vec{\xi} = r\vec{\xi}$$

Eigenvalues are easy to find since this is a triangular matrix $r = 3,\ r = 1$

$r_1 = 3$ Find eigenvector $\begin{bmatrix} 3-3 & -1 \\ 0 & 1-3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \vec{0}$

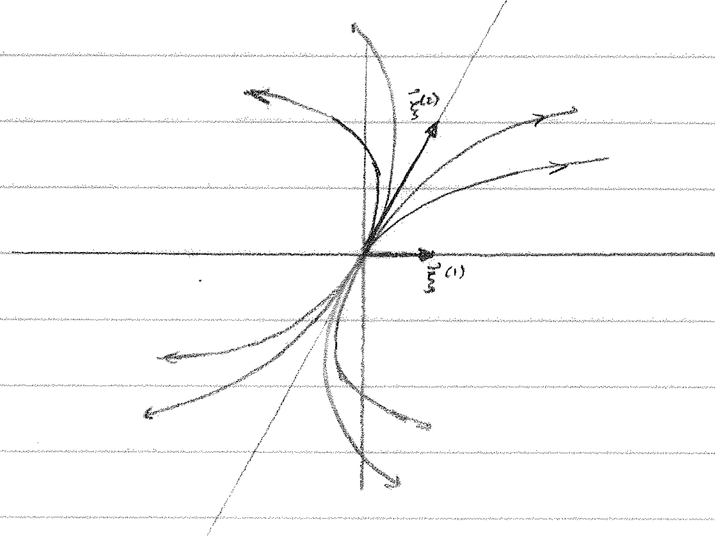
$0\xi_1 - \xi_2 = 0$ $\quad \vec{\xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$r_2 = 1$ $\quad \begin{bmatrix} 3-1 & -1 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \vec{0}$

$2\xi_1 - \xi_2 = 0$ $\quad \vec{\xi} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}$ $\quad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t}$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t}$$



Ex # 4 $\qquad \vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$

Find eigenvalues: $\quad \det \begin{bmatrix} 1-r & 1 \\ 4 & -2-r \end{bmatrix} = 0$

$$(1-r)(-2-r) - 4 = 0$$
$$-2 + r + r^2 - 4 = 0$$
$$r^2 + r - 6 = 0$$
$$(r-2)(r+3) = 0 \qquad r = 2, \quad r = -3$$

Find eigenvectors:

$$r = 2 \qquad \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \xi_1 = \xi_2 \rightarrow \vec{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$r = -3 \qquad \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 4\xi_1 = -\xi_2 \rightarrow \vec{\xi} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Verify linear independence

$$\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \qquad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \begin{vmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{vmatrix} = -5e^{-t} \neq 0$$

Construct General Solution

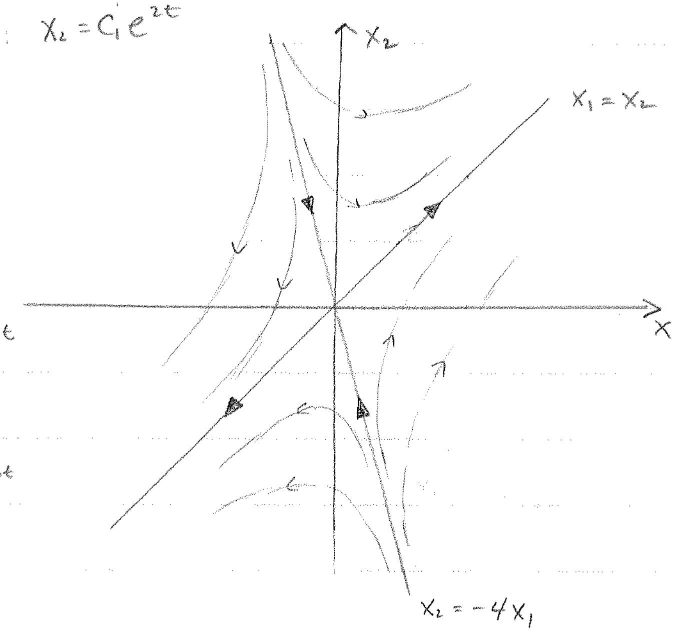
$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

Sketch a few trajectories

If $C_2 = 0$ then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

or $x_1 = c_1 e^{2t}$; $x_2 = c_1 e^{2t}$

$\rightarrow$ $x_2 = x_1$



If $C_1 = 0$ then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$

or

$x_1 = c_2 e^{-3t}$ $x_2 = -4c_2 e^{-3t}$

$\rightarrow$ $x_1 = \frac{x_2}{-4}$

$x_2 = -4x_1$

arrows show direction of motion as t increases

Discuss solutions as $t \to \infty$

If $c_1 = 0$ then $\vec{x} \to \vec{0}$ as $t \to \infty$

otherwise $x_1, x_2 \to +\infty$ if $c_1 > 0$

$x_1, x_2 \to -\infty$ if $c_1 < 0$

The origin is a <u>saddle point</u>.

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Ex #12 $\vec{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \vec{x}$

Here the matrix is symmetric (Hermitian) so we have

1. real eigenvalues
2. 3 lin. indep. eigenvectors

$$\det \begin{bmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{bmatrix} = 0 \rightarrow \quad r=4, \ r=1, \ r=-1$$

$r=4$: $\begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} -3 & 1 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} \xi_1 = \xi_3 \\ \xi_2 = \xi_3 \\ \xi_3 = \xi_3 \end{array} \rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Octave
$[P\ D] = \mathrm{eig}(A)$
normalized eigenvectors
eigenvalues on diagonal

$r=1$: $\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightarrow \begin{array}{l} \xi_1 = \xi_3 \\ \xi_2 = -2\xi_3 \\ \xi_3 = \xi_3 \end{array} \rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$r=-1$: $\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightarrow \begin{array}{l} \xi_1 = -\xi_3 \\ \xi_2 = 0 \\ \xi_3 = \xi_3 \end{array} \rightarrow \vec{\xi}^{(3)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

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So the general solution is

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

Suppose that this problem had the initial condition

$$\vec{X}(0) = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} \quad \text{find } c_1, c_2, c_3$$

at $t=0$ we have $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$

or

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 1 & -2 & 0 & 3 \\ 1 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & -3 & 1 & 6 \\ 0 & 0 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

So

$$\vec{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} e^{t} + \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} e^{-t}$$