7.4 Basic Theory of Systems of 1st Order Linear Equations

Our general eq. is $\vec{x}' = P\vec{x} + \vec{g}$

where $\vec{x} = \vec{x}(t)$

$P = P(t)$

$\vec{g} = \vec{g}(t)$

As in the single eq. case, we will begin by considering the case when $\vec{g}(t) = \vec{0}$ so

$$\vec{x}' = P\vec{x} \qquad (*)$$

<u>Thm 7.4.1</u> If $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions of $(*)$ then $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is also a solution $\forall\, c_1, c_2$.

<u>Thm 7.4.2</u> If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are linearly independent solutions of $(*)$ for each point in $\alpha < t < \beta$ then each solution $\vec{x} = \vec{\phi}(t)$ of $(*)$ can be expressed as

$$\vec{\phi}(t) = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \dots + c_n\vec{x}^{(n)}$$

in exactly one way.

<u>Thm 7.4.3</u> If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are solutions of $(*)$ on $\alpha < t < \beta$ then in this interval $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}]$ is either identically zero or else never vanishes.

#7 $\vec{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ $\vec{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

a) Compute the Wronskian of $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$

$$W = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = t^2e^t - 2te^t = te^t(t-2)$$

b) These solutions are linearly independent on

$\boxed{t<0, \quad 0<t<2, \quad 2<t}$

since at $t=0$ and at $t=2$ the Wronskian is zero

c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$?

One or more coefficients must be discontinuous at $t=0$ and at $t=2$.

d) Need to find A so that $\vec{x}' = A\vec{x}$ for all $\vec{x} = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$.

$$\begin{bmatrix} 2t & e^t \\ 2 & e^t \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t^2 & e^t \\ 2t & e^t \end{bmatrix}$$

$at^2 + b(2t) = 2t$ $\quad (a+b)e^t = e^t$

$ct^2 + d(2t) = 2$ $\quad (c+d)e^t = e^t$

$at + 2b = 2$ $\quad a+b = 1$

$ct + 2d = 2/t$ $\quad c+d = 1$

#7 (continued)

$at+2b=2 \qquad a+b=1$

$b=1-a$

$at+2(1-a)=2$

$at+2-2a=2$

$a(t-2)=0 \qquad a=0,\ b=1$

$ct+2d=2/t \qquad c+d=1$

$d=1-c$

$ct+2(1-c)=2/t$

$ct+2-2c=2/t$

$c(t-2)=2/t-2$

$c=\frac{2-2t}{t(t-2)} \qquad c=2\frac{1-t}{t(t-2)}$

$d=1-2\frac{1-t}{t(t-2)}$

$=\frac{t(t-2)-2+2t}{t(t-2)}$

$=\frac{t^2-2}{t(t-2)}$

$$A=\begin{bmatrix} 0 & 1 \\ \frac{2-2t}{t(t-2)} & \frac{t^2-2}{t(t-2)} \end{bmatrix}$$