7.3 Systems of Linear Algebraic Equations: Linear Independence, Eigenvalues, Eigenvectors

The $n \times n$ matrix A is <u>nonsingular</u> if it is invertible, otherwise it is <u>singular</u>

The system $A\vec{x}=\vec{b}$ has a unique solution iff A is nonsingular, in which case

$$\vec{x} = A^{-1}\vec{b}$$

$A\vec{x}=\vec{b}$ is a <u>nonhomogeneous</u> eq. unless $\vec{b}=\vec{0}$.

$A\vec{x}=\vec{0}$ is a <u>homogeneous</u> system

A nonsingular: $A\vec{x}=\vec{0}$ has only trivial solution $\vec{x}=\vec{0}$
$A\vec{x}=\vec{b}$ has unique solution

A singular: $A\vec{x}=\vec{0}$ has infinitely many solutions
$A\vec{x}=\vec{b}$ may have no solution <u>or</u> infinitely many.

no solution unless $\vec{b}$ is in the range of A, i.e. unless $\vec{b}$ is in the span of the columns of A, i.e. unless $\vec{b}$ is a linear combination of the columns of A, etc.

omit

If $(\vec{b},\vec{y})=0 \ \forall \ \vec{y}$ s.t $A^*\vec{y}=\vec{0}$ then $A\vec{x}=\vec{b}$ has solutions when A is singular

$[\vec{b}^T\vec{y}=0 \ \forall \ \vec{y}$ s.t $A^T\vec{y}=0]$ then there will be infinitely many solns.

Solutions of $A\vec{x}=\vec{b}$ when A is singular

These are of the form $\vec{x} = \vec{x}^{\,0} + \vec{\xi}$ where $A\vec{x}^{\,0} = \vec{b}$ and $A\vec{\xi} = \vec{0}$. $\vec{\xi}$ is <u>any</u> of the infinite number of solutions of $A\vec{\xi} = \vec{0}$. $\vec{x}^{\,0}$ is a particular solution

We can solve $A\vec{x} = \vec{b}$ (or determine it has a solution) by performing row operations on the <u>augmented matrix</u>

$$[A \mid \vec{b}]$$

<u>Ex</u>

$$\begin{bmatrix} 1 & -1 & 4 \\ -3 & 6 & -10 \\ -2 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ -1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 6 \\ -3 & 6 & -10 & -5 \\ -2 & 5 & -7 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & 6 \\ 0 & 3 & 2 & 13 \\ 0 & 3 & 1 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & 6 \\ 0 & 3 & 2 & 13 \\ 0 & 0 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{So} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

<u>Ex</u>

$$\begin{bmatrix} 3 & 6 & 3 \\ 0 & 2 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -16 \\ 32 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 3 & 6 & 3 & 0 \\ 0 & 2 & -4 & -16 \\ 0 & -4 & 8 & 32 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 1 & -2 & -8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

if this was non-zero, there would be no solution.

row of all zeros

$\rightarrow$ infinitely many solutions

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to

$$\begin{aligned} x_1 \quad\quad + 5x_3 &= 16 \\ x_2 - 2x_3 &= -8 \end{aligned}$$

if $x_3 = \alpha$ then

$$\begin{aligned} x_1 &= 16 - 5\alpha \\ x_2 &= -8 + 2\alpha \\ x_3 &= \quad\quad 1\alpha \end{aligned}$$

so

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 16 \\ -8 \\ 0 \end{bmatrix}}_{\text{particular soln.}} + \alpha \underbrace{\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}}_{\text{soln of homogeneous eq.}}$$

A set of vectors $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are linearly independent if $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_n\vec{x}^{(n)} = 0$ iff $c_1 = c_2 = \cdots = c_n = 0$, otherwise they are lin. dependent.

If $\underline{X} = [\vec{x}^{(1)} \ \vec{x}^{(2)} \cdots \vec{x}^{(n)}]$ is a matrix whose columns are the $\vec{x}^{(i)}$ then

$$\det \underline{X} \neq 0 \iff \vec{x}^{(i)} \text{ are lin. independent}$$

Consider $A\vec{x} = \lambda\vec{x}$. If λ and $\vec{x} \neq \vec{0}$ can be found for which this is true then λ is an eigenvalue of A and $\vec{x}$ is the corresponding eigenvector.

<u>Ex</u> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

A must satisfy $A\vec{x} = \lambda \vec{x}$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$A\vec{x} - \lambda I \vec{x} = \vec{0}$$
$$(A - \lambda I)\vec{x} = \vec{0}$$

So if any eigenvectors are to be found ($\vec{x} = \vec{0}$ is <u>never</u> an eigenvector) then $A - \lambda I$ must be singular so

$$\det(A - \lambda I) = 0 \qquad \text{Characteristic Eq.}$$

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - 9 = 0$$
$$= -12 + 4\lambda + \lambda^2 - 9 = 0$$

So

$$\lambda^2 + 4\lambda - 21 = 0$$
$$(\lambda - 3)(\lambda + 7) = 0 \rightarrow \boxed{\lambda = 3, \ \lambda = -7}$$

To find the eigenvectors, solve $(A - \lambda I)\vec{x} = 0$ for each λ.

1. $\lambda = 3 \rightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -9 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} x_1 - 3x_2 = 0 \\ x_2 = \alpha \end{array} \rightarrow \boxed{\vec{x} = \alpha \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$

2. $\lambda = -7 \rightarrow \left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} 3x_1 + x_2 = 0 \\ x_1 = \alpha \end{array} \rightarrow \boxed{\vec{x} = \alpha \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$

An $n\times n$ matrix is <u>Hermitian</u> or <u>self-adjoint</u> if $A^* = A$.
If A has only real values then this is the same as symmetric: $A^T = A$.

If A is Hermitian (or real and symmetric) then

1. A has real eigenvalues
2. A has n - linearly independent eigenvectors
3. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
   i.e. $(\vec{x}^{(1)}, \vec{x}^{(2)}) = 0$
4. If λ is not simple (i.e. λ has multiplicity > 1) then all associated eigenvectors can be chosen to be orthogonal.

A matrix A is diagonalizable if $\exists$ a matrix T such that

$$A = TDT^{-1}$$

where D is a diagonal matrix. In this case, the diagonal elements of D are the eigenvalues of A and the columns of T are the associated eigenvectors.

Special case of a similarity transform. A is <u>similar</u> to B if $\exists$ a matrix T s.t. $A = TBT^{-1}$. If this is the case then A and B have the same eigenvalues.