7.2 Review of Matrices.

The matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ has m rows

and n columns. Thus it is an $\overset{\text{rows}}{m} \times \overset{\text{columns}}{n}$ matrix

The entries a_{ij} are called the <u>entries</u> in A and may be either real or complex.

If $A = (a_{ij})$ then $A^T = (a_{ji})$ is the <u>transpose</u> of A.

We will use $\bar{A} = (\bar{a}_{ij})$ to denote the matrix of complex conjugates of the a_{ij}. It is called the <u>conjugate</u> of A.

<u>Ex</u> $A = \begin{bmatrix} 2+i & 3-2i \\ 4 & 6 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2+i & 4 \\ 3-2i & 6 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 2-i & 3+2i \\ 4 & 6 \end{bmatrix}$ $\quad A^* = \begin{bmatrix} 2-i & 4 \\ 3+2i & 6 \end{bmatrix}$

where $A^* = \bar{A}^T$ is the <u>adjoint</u> of A.

Properties

1. $A = B$ if $a_{ij} = b_{ij}$ $\quad i = 1, \ldots, m \quad j = 1, \ldots, n$
2. $C = A + B$ if $c_{ij} = a_{ij} + b_{ij}$ $\quad$ " $\quad$ "
3. $A + B = B + A$
4. $(A+B)+C = A+(B+C)$

5. $B = cA$ if $b_{ij} = c a_{ij}$ $i=1,\dots,m;\ j=1,\dots,n$

6. $c(A+B) = cA + cB$

7. $(c+d)A = cA + dA$

8. $C = AB$ if $c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$, A $m \times r$, B $r \times n$, C $m \times n$

9. $A(B+C) = AB + AC$

10. $(A+B)C = AC + BC$

11. $(AB)C = A(BC)$

12. $AB \neq BA$ (in general)

A is $m \times r$, B is $r \times n$ then AB is $m \times n$. BA is defined only if $n = m$ and even then it will generally not be the same as AB.

Vectors

We will denote $n \times 1$ matrices by lowercase letters topped with an $\rightarrow$ and call these <u>vectors.</u> If $\vec{x}$ is $n \times 1$ then $\vec{x}^T$ is $1 \times n$.

$\vec{x}\,\vec{y}^T$ is an $n \times n$ matrix, while $\vec{x}^T \vec{y}$ is a scalar

The <u>inner</u> or <u>scalar product</u> of $\vec{x}$ and $\vec{y}$ is $(\vec{x}, \vec{y})$ and is equal to

$$(\vec{x}, \vec{y}) = \vec{x}^T \overline{\vec{y}}$$

$$= \sum_{i=1}^{n} x_i \overline{y}_i$$

denotes complex conj.

Properties of the inner Product

1. $(\vec{x},\vec{y}) = \overline{(\vec{y},\vec{x})}$
2. $(\vec{x},\vec{y}+\vec{z}) = (\vec{x},\vec{y}) + (\vec{x},\vec{z})$
3. $(a\vec{x},\vec{y}) = a(\vec{x},\vec{y})$
4. $(\vec{x}, b\vec{y}) = \bar{b}(\vec{x},\vec{y})$

The complex conjugate of $a+ib$ is $\overline{a+ib}$ and is equal to $a-ib$.

$$(\vec{x},\vec{x}) = \vec{x}^{T}\bar{\vec{x}} = \sum_{i=1}^{n} |x_i|^2 .$$

The magnitude of $\vec{x}$ is $\|\vec{x}\| = \sqrt{(\vec{x},\vec{x})}$ (ℓ_2 norm)

If $(\vec{x},\vec{y}) = 0$ then $\vec{x}$ and $\vec{y}$ are orthogonal

A $(n \times n)$ matrix is invertible if $\exists$ an $n \times n$ matrix A^{-1} s.t.

$$AA^{-1} = A^{-1}A = I \qquad (A \text{ is } \underline{\text{nonsingular}})$$

where I is the $n \times n$ identity matrix (1's on diagonal, 0's elsewhere)

One way to invert A (find A^{-1}) is to <u>row-reduce</u> A to I, while performing the same row operations on I.

Row reduction:
1. interchange any 2 rows
2. mult. a row by a nonzero scalar
3. add a mult. of a row to any other row.

Ex $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ Find A^{-1}

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 0 & \frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right]$$

row equivalent $\quad -\frac{4}{3}R_1 + R_2 \Rightarrow R_2$

$$\sim \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 0 & 1 & -4 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 0 & 21 & -15 \\ 0 & 1 & -4 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 7 & -5 \\ 0 & 1 & -4 & 3 \end{array}\right]$$

$3R_2 \Rightarrow R_2 \qquad -5R_2 + R_1 \Rightarrow R_1 \qquad \frac{1}{3}R_1 \Rightarrow R_1$

So $A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$

If A is 2×2 then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If A is invertible it is <u>nonsingular</u>. If an $n \times n$ matrix is not invertible it is <u>singular</u>.

Matrix Functions

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ & \ddots & \\ a_{m1}(t) & & a_{mn}(t) \end{bmatrix}$$ entries are functions of t.

- $A(t)$ is continuous on $\alpha < t < \beta$ if each $a_{ij}(t)$ is continuous on that interval
- Operations like differentiation behave as expected

Ex

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B \quad \text{(cannot change order)}$$