7.1 Introduction (Systems of First Order Linear Equations)

A system of n first order linear differential eq. has the form

$$\left.\begin{aligned}
x_1' &= P_{11}(t)\,x_1 + P_{12}(t)\,x_2 + \cdots + P_{1n}(t)\,x_n + g_1(t) \\
x_2' &= P_{21}(t)\,x_1 + P_{22}(t)\,x_2 + \cdots + P_{2n}(t)\,x_n + g_2(t) \\
&\ \vdots \\
x_n' &= P_{n1}(t)\,x_1 + P_{n2}(t)\,x_2 + \cdots + P_{nn}(t)\,x_n + g_n(t)
\end{aligned}\right\} \quad (*)$$

where $P_{ij}(t)$ and $g_i(t)$ are functions of t as are $x_i(t)$.

We can write this in the matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{22} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$

If $\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{g}(t) = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$ and $P(t)$ is the matrix above,

Then

$$\vec{x}'(t) = P(t)\,\vec{x}(t) + \vec{g}(t)$$

or just

$$\vec{x}' = P\vec{x} + \vec{g}$$

If $\vec{g} = \vec{0}$ then the system is homogeneous, otherwise it is nonhomogeneous.

Thm 7.1.2

If p_{ij} and g_i, $i,j = 1, \dots, n$, are continuous on an open interval $I: \alpha < t < \beta$ then $\exists$ a unique solution $x_1 = \phi_1(t), \dots, x_n = \phi_n(t)$ of the system $(*)$ that also satisfies the initial conditions $x_1(t_0) = x_1^0, \dots, x_n(t_0) = x_n^0$ and this solution exists throughout I.

Ex

Reduction of higher order eq.

Consider $x'' + p x' + q x = g$

If $x_1 = x$, $x_2 = x' = x_1'$ then $x'' = x_2'$ and

$$\begin{aligned} x_1' &= x_2 \\ x_2' + p x_2 + q x_1 &= g \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1' &= \phantom{-q x_1 - {}} x_2 \\ x_2' &= -q x_1 - p x_2 + g \end{aligned}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$x''' - 3x' + x = 2 \quad \rightarrow \quad \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' - 3x_2 + x_1 &= 2 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1' &= + x_2 \\ x_2' &= + x_3 \\ x_3' &= -x_1 + 3x_2 + 2 \end{aligned}$$

$x_1 = x$

$x_2 = x' = x_1'$

$x_3 = x'' = x_2'$