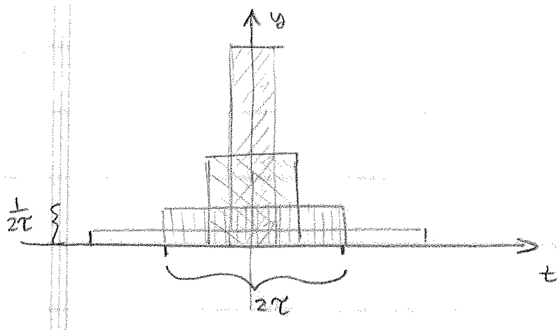
6.5 Impulse Functions

Construct a function with the properties:

1) centered about origin
2) identically zero outside $(-\tau, \tau)$
3) Total area under curve is one.



$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & t \le -\tau \text{ or } \tau \le t \end{cases}$$

is one such function ...

Since $\displaystyle\int_{-\infty}^{\infty} d_\tau(t)\, dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}\, dt = \frac{1}{2\tau}\, t \Big|_{-\tau}^{\tau} = \frac{1}{2\tau}(\tau+\tau) = 1$

independent of τ we find that $\displaystyle\lim_{\tau \to 0} \int_{-\infty}^{\infty} \frac{1}{2\tau}\, dt = \underline{1}$

The Dirac Delta $\delta(t)$ is a <u>generalized function</u> with the properties

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

one way to get this is

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t)$$

Thus, $\delta(t - t_0)$ is a <u>unit impulse function</u>, acting at $t = t_0$

Now, let's find $\mathcal{L}\{\delta(t-t_0)\}$. We define this to be

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{\tau\to 0} \mathcal{L}\{d_\tau(t-t_0)\}$$

$$= \lim_{\tau\to 0} \int_0^\infty e^{-st} d_\tau(t-t_0)\,dt$$

$$= \lim_{\tau\to 0} \int_{t_0-\tau}^{t_0+\tau} e^{-st} \frac{1}{2\tau}\,dt$$

$$= \lim_{\tau\to 0} \frac{1}{2\tau} \frac{e^{-st}}{-s} \Bigg|_{t_0-\tau}^{t_0+\tau}$$

$$= \lim_{\tau\to 0} -\frac{1}{2s\tau}\left[e^{-s(t_0+\tau)} - e^{-s(t_0-\tau)}\right]$$

$$= \lim_{\tau\to 0} -\frac{1}{2s\tau} e^{-st_0}\left[e^{-s\tau} - e^{s\tau}\right]$$

$$= \lim_{\tau\to 0} e^{-st_0} \frac{e^{s\tau}-e^{-s\tau}}{2s\tau} = \lim_{\tau\to 0} e^{-st_0} \frac{\sinh s\tau}{s\tau}$$

$$= e^{-st_0}$$

So $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$.

[In general $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$]

6.5 Impulse Function (short form) 1 and 2

Let $d_\tau(t) = \begin{cases} 1/2\tau & -\tau < t < \tau \\ 0 & t \le -\tau \text{ or } \tau \le t \end{cases}$

area under curve is 1

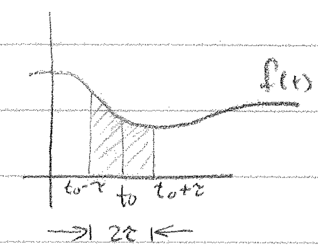
Define <u>Dirac Delta</u> $\delta(t) = \lim_{\tau \to 0} d_\tau(t)$

This is an <u>impulse function</u>, or <u>unit impulse function</u>

$$\int_{-\infty}^{\infty} \delta(t-t_0)\, f(t)\, dt = \lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t-t_0)\, f(t)\, dt$$

$$= \lim_{\tau \to 0} \int_{-\tau+t_0}^{\tau+t_0} \frac{1}{2\tau} f(t)\, dt$$

$$= \lim_{\tau \to 0} \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} f(t)\, dt$$



as $\tau \to 0$ this area approaches

$f(t_0) \cdot 2\tau$

height width

or $\lim_{\tau \to 0} \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} f(t)\, dt = \lim_{\tau \to 0} \frac{F(t_0+\tau) - F(t_0-\tau)}{2\tau} = F'(t_0) = f(t_0)$

So our integral becomes $= \lim_{\tau \to 0} \frac{1}{2\tau} f(t_0) \cdot (2\tau)$

$= f(t_0)$

$$\therefore \int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0)$$

So $\boxed{\mathcal{L}\{\delta(t-t_0)\} = \int_0^{\infty} e^{-st} \delta(t-t_0)\, dt = e^{-st_0}}$

<u>Ex</u> Solve $\begin{cases} y'' + 2y' + y = \delta(t) + u_{2\pi}(t) \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$

So $$s^2 Y(s) - s y(0) - y'(0) + 2\left[sY(s) - y(0)\right] + Y(s)$$

$$= \mathcal{L}\{\delta(t)\} + \mathcal{L}\{u_{2\pi}(t)\}$$

$$(s^2 + 2s + 1)Y(s) - 1 = e^{-s \cdot 0} + e^{-2\pi s} \cdot \frac{1}{s} = 1 + \frac{e^{-2\pi s}}{s}$$

$$Y(s) = \frac{2 + \frac{e^{-2\pi s}}{s}}{s^2 + 2s + 1} = \frac{2}{(s+1)^2} + \frac{e^{-2\pi s}}{s(s+1)^2}$$

Part-by-part:

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2}\right\} = 2 \cdot e^{-t} \cdot t = 2te^{-t}$$

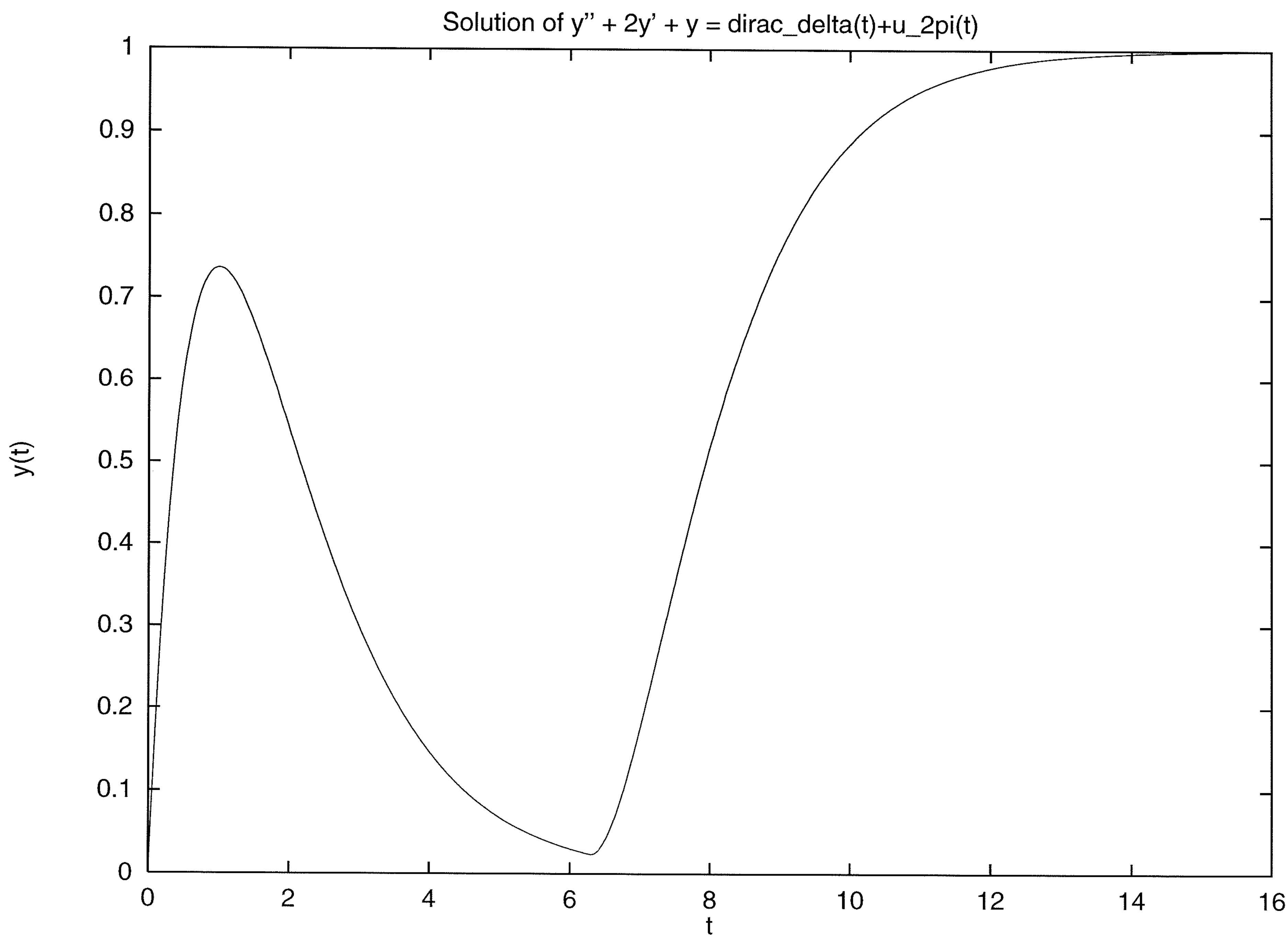
$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \rightarrow \quad A = 1, \; B = C = -1$$

so
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right\} = 1 - e^{-t} \cdot 1 - e^{-t} \cdot t$$
$$= 1 - e^{-t}(1+t)$$

so...
$$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s(s+1)^2}\right\} = u_{2\pi}(t)\left[1 - e^{-(t-2\pi)}(1 + (t - 2\pi))\right]$$

Thus

$$y(t) = 2te^{-t} + u_{2\pi}(t)\left[1 - e^{-(t-2\pi)}(t - 2\pi + 1)\right]$$



Solution of y'' + 2y' + y = dirac_delta(t)+u_2pi(t)
y(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
2
4
6
8
10
12
14
16

Ex $\begin{cases} y'' - y = \delta(t - \frac{\pi}{2})\sin t \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$

$$s^2 Y(s) - s y(0) - y'(0) - Y(s) = \mathcal{L}\{\delta(t-\tfrac{\pi}{2})\sin t\}$$

but $\mathcal{L}\{\delta(t-\frac{\pi}{2})\sin t\} = \int_0^\infty \delta(t-\frac{\pi}{2}) e^{-st} \sin t \, dt$

$$= e^{-s\frac{\pi}{2}} \sin\tfrac{\pi}{2} \qquad \text{since } \int_0^\infty \delta(t-t_0) f(t)\,dt = f(t_0)$$

$$= e^{-\frac{\pi}{2}s}$$

So

$$(s^2-1)\,Y(s) = e^{-\frac{\pi}{2}s} \rightarrow Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s^2-1}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$ so

$$y(t) = u_{\frac{\pi}{2}}(t)\,\sinh(t-\tfrac{\pi}{2})$$

Notice

$$\mathcal{L}\{\delta(t-t_0) f(t)\} = \int_0^\infty e^{-st}\,\delta(t-t_0) f(t)\,dt$$

$$= \int_{-\infty}^\infty e^{-st}\,\delta(t-t_0) f(t)\,dt \qquad \text{if } t_0 \geq 0$$

$$= e^{-st_0} f(t_0)$$

So

$$\mathcal{L}\{\delta(t-t_0) f(t)\} = e^{-st_0} f(t_0)$$