6.4 Differential Equations with Discontinuous Forcing Functions

Ex #26 section 2.2

$$y' + 2y = g(t) \qquad y(0) = 0 \qquad g(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

So $y' + 2y = 1 - u_1(t)$

$$sY(s) - y(0) + 2Y(s) = \mathcal{L}\{1\} - \mathcal{L}\{u_1(t)\}$$
$$= \frac{1}{s} - e^{-s}\frac{1}{s} = (1 - e^{-s})\frac{1}{s}$$

$$(s+2)Y(s) = \frac{1}{s}(1 - e^{-s}) \quad \rightarrow \quad Y(s) = \frac{1 - e^{-s}}{s(s+2)}$$

Let $H(s) = \frac{1}{s(s+2)}$ so $Y(s) = (1 - e^{-s})H(s)$

if $\mathcal{L}\{h(t)\} = H(s)$ then $y(t) = h(t) - u_1(t)\,h(t-1)$

$$H(s) = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \quad \rightarrow \quad A = \frac{1}{2}, \; B = -\frac{1}{2}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$$

$\therefore\; h(t) = \frac{1}{2}(1) - \frac{1}{2}e^{-2t}$

So
$$y(t) = \frac{1}{2}(1 - e^{-2t}) - u_1(t)\frac{1}{2}(1 - e^{-2(t-1)})$$
$$= \frac{1}{2}(1 - e^{-2t}) - \frac{1}{2}u_1(t)(1 - e^{-2t}e^{2})$$

$$= \begin{cases} \frac{1}{2}(1 - e^{-2t}) & 0 \le t \le 1 \\ \frac{1}{2}e^{-2t}(e^{2} - 1) & t > 1 \end{cases}$$

6.4

#8 $y'' + y' + \frac{5}{4}y = t - u_{\frac{\pi}{2}}(t)\,(t-\frac{\pi}{2})$ $\quad y(0)=0 \quad y'(0)=0$

$$s^2\,Y(s) - s\,y(0) - y'(0) + s\,Y(s) - y(0) + \frac{5}{4}Y(s) = \frac{1}{s^2} - e^{-\frac{\pi}{2}s}\,\frac{1}{s^2}$$

$$\left(s^2+s+\frac{5}{4}\right)Y(s) = \frac{1}{s^2}\left(1-e^{-\frac{\pi}{2}s}\right)$$

$$Y(s) = \left(1-e^{-\frac{\pi}{2}s}\right)\frac{1}{s^2(s^2+s+5/4)} = \left(1-e^{-\frac{\pi}{2}s}\right)\frac{1}{s^2[(s+\frac{1}{2})^2+1]}$$

$$H(s) = \frac{1}{s^2[(s+\frac{1}{2})^2+1]} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{(s+\frac{1}{2})^2+1}$$

$A = -\frac{16}{25}$, $B = \frac{20}{25}$, $C = \frac{16}{25}$, $D = -\frac{4}{25}$ $\quad$ * (see addundum) page 2a

$$H(s) = -\frac{16}{25}\frac{1}{s} + \frac{20}{25}\frac{1}{s^2} + \frac{4}{25}\frac{4s-1}{(s+\frac{1}{2})^2+1} = -\frac{16}{25}\frac{1}{s} + \frac{20}{25}\frac{1}{s^2} + \frac{16}{25}\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1} - \frac{4}{25}\frac{3}{(s+\frac{1}{2})^2+1}$$

$$h(t) = -\frac{16}{25} + \frac{20}{25}t + \frac{16}{25}e^{-\frac{1}{2}t}\cos t - \frac{12}{25}e^{-\frac{1}{2}t}\sin t$$
$$= \frac{4}{25}\left[-4+5t+e^{-\frac{1}{2}t}(4\cos t - 3\sin t)\right]$$

now find $y(t)$

$$y(t) = \mathcal{L}\{(1-e^{-\frac{\pi}{2}s})H(s)\} = h(t) - u_{\frac{\pi}{2}}(t)\,h(t-\frac{\pi}{2})$$

So

$$y(t) = h(t) - u_{\frac{\pi}{2}}(t)\,h(t-\frac{\pi}{2})$$

where $h(t) = \frac{4}{25}\left[-4+5t+e^{-\frac{1}{2}t}(4\cos t - 3\sin t)\right]$

Find the partial fraction decomposition of $\dfrac{1}{s^2[(s+\frac{1}{2})^2+1]}$

$$\frac{1}{s^2[(s+\frac{1}{2})^2+1]} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{(s+\frac{1}{2})^2+1}$$

$$\begin{aligned}
1 &= As[(s+\tfrac{1}{2})^2+1] + B[(s+\tfrac{1}{2})^2+1] + (Cs+D)s^2 \\
&= As(s^2+s+\tfrac{5}{4}) + B(s^2+s+\tfrac{5}{4}) + Cs^3+Ds^2 \\
&= As^3+As^2+\tfrac{5}{4}As + Bs^2+Bs+\tfrac{5}{4}B + Cs^3+Ds^2 \\
&= s^3(A+C) + s^2(A+B+D) + s(\tfrac{5}{4}A+B) + \tfrac{5}{4}B
\end{aligned}$$

Matching coefficients on both sides of the equation we have

$$\begin{array}{lcl}
A+C=0 & \text{so} & B = 4/5 \\
A+B+D=0 & & A = -16/25 \\
\frac{5}{4}A+B=0 & & C = 16/25 \\
\frac{5}{4}B=1 & & D = -4/25
\end{array}$$

Therefore

$$\frac{1}{s^2[(s+\frac{1}{2})^2+1]} = \frac{-16}{25s} + \frac{20}{25s^2} + \frac{1}{25}\,\frac{16s-4}{(s+\frac{1}{2})^2+1}$$

6.4

In general, if $y'' + p(t)\,y' + q(t)\,y = g(t)$

We require $p(t)$ and $q(t)$ to be continuous on some interval but only that $g(t)$ be piecewise continuous.

Then if $y(t) = \psi(t)$ is a solution, $\psi(t)$ and $\psi'(t)$ will be continuous on the interval but $\psi''(t)$ will be discontinuous at the same points $g(t)$ is.