6.2 Solutions of Initial Value Problems

$\mathcal{L}\{f'(t)\} = ?$

$$\mathcal{L}\{f'(t)\} = \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\,dt$$

$$I = \int_0^b e^{-st} f'(t)\,dt$$

We will assume that the function $f'(t)$ is cont. from 0 to A – result is the same for piecewise cont. functions

$$u = e^{-st} \qquad dv = f'(t)\,dt$$
$$du = -se^{-st}dt \qquad v = f(t)$$

$$I = e^{-st} f(t)\Big|_0^b + \int_0^b s e^{-st} f(t)\,dt$$

$$= e^{-sb} f(b) - e^{-s(0)} f(0) + s\int_0^b e^{-st} f(t)\,dt$$

$$\lim_{b\to\infty} I = 0 \cdot f(b) - f(0) + s\,\mathcal{L}\{f(t)\}$$

So $\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$
$$= s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$
$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

in general

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

Thm 6.2.1

Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. If $\exists$ K, a, M such that $|f(t)| \le Ke^{at}$ for $t \ge M$ then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

Ex $\quad y'' - y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = -1$

$$\begin{aligned}
\mathcal{L}\{y''\} &= s^2 Y(s) - s\,y(0) - y'(0) &&= s^2 Y(s) - s + 1 \\
\mathcal{L}\{y'\} &= s\,Y(s) - y(0) &&= s\,Y(s) - 1 \\
\mathcal{L}\{y\} &= Y(s) &&= Y(s)
\end{aligned}$$

So $\quad y'' - y' - 6y = s^2 Y(s) - s + 1 - s\,Y(s) + 1 - 6\,Y(s) = 0$

$$= (s^2 - s - 6)\,Y(s) - s + 2 = 0$$

(characteristic eq $s^2 - s - 6 = 0$)

So $\quad Y(s) = \dfrac{s-2}{s^2 - s - 6} = \dfrac{s-2}{(s-3)(s+2)}$

Now the equation is "solved". Key here is that The Laplace transforms of derivatives of functions are transformed into algebraic combinations of the unknown function $Y(s)$ and known values $y^{(i)}(0)$, $i = 0, \dots, n-1$ (initial cond).

The problem now is to figure out what function has $Y(s)$ as its Laplace Transform.

$$Y(s) = \frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} \qquad \left(\text{Partial fractions}\right)$$

$$A = \frac{1}{5} \qquad B = \frac{-4}{-5} = \frac{4}{5}$$

$$Y(s) = \frac{1}{5}\frac{1}{s-3} + \frac{4}{5}\frac{1}{s+2}$$

$\downarrow \qquad \downarrow$ inverse transforms

$$y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$ using table in section 6.2

so $\boxed{y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}}$

Ex $y^{iv} - 4y = 0 \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - 4Y(s) = 0$$

$$(s^4 - 4)Y(s) - 0 - 0 - s(-2) - 0 = 0$$

$$(s^4 - 4)Y(s) + 2s = 0$$

$$Y(s) = \frac{-2s}{s^4 - 4} = \frac{-2s}{(s^2+2)(s^2-2)}$$

$$\frac{-2s}{(s^2+2)(s^2-2)} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2-2}$$

$$-2s = (As+B)(s^2-2) + (Cs+D)(s^2+2)$$

or

$s = \sqrt{2} \rightarrow \quad -2\sqrt{2} = 4(C\sqrt{2} + D)$

$s = -\sqrt{2} \rightarrow \quad 2\sqrt{2} = 4(-C\sqrt{2} + D)$

adding yields $D = 0$

subtracting $-4\sqrt{2} = 8C\sqrt{2} \qquad C = -\frac{1}{2}$

$s = 0 \rightarrow \quad 0 = B(-2) \quad$ so $\quad B = 0 \qquad A = \frac{1}{2}$

$s = 1 \rightarrow \quad -2 = A(-1) + (-\frac{1}{2})(3) \qquad B = 0$

$-4 = -2A - 3 \qquad C = -\frac{1}{2}$

$-1 = -2A \qquad A = \frac{1}{2} \qquad D = 0$

So $Y(s) = \frac{1}{2}\frac{s}{s^2+2} - \frac{1}{2}\frac{s}{s^2-2}$

$$\boxed{y(t) = \frac{1}{2}\cos\sqrt{2}\,t - \frac{1}{2}\cosh\sqrt{2}\,t}$$

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$-2s = As^3 + Bs^2 - 2As - 2B + Cs^3 + Ds^2 + 2Cs + 2D$

$-2s = (A+C)s^3 + (B+D)s^2 + (2C-2A)s + (2D-2B)$

$\therefore \quad A + C = 0 \qquad 2C - 2A = -2$

$B + D = 0 \qquad 2D - 2B = 0$

$\left.\begin{array}{l} A + C = 0 \\ A - C = 1 \end{array}\right\} \Rightarrow \begin{array}{l} A = \frac{1}{2} \\ C = -\frac{1}{2} \end{array} \qquad \left.\begin{array}{l} B + D = 0 \\ D - B = 0 \end{array}\right\} \begin{array}{l} B = 0 \\ D = 0 \end{array}$

$\therefore \quad Y(s) = \frac{1}{2}\frac{s}{s^2+2} - \frac{1}{2}\frac{s}{s^2-2}$

$$\boxed{y(t) = \frac{1}{2}\cos\sqrt{2}\,t - \frac{1}{2}\cosh\sqrt{2}\,t}$$