6.1 Definition of the Laplace Transform

Recall from calculus how we evaluate improper integrals.

$$\int_a^\infty f(t)\,dt = \lim_{b\to\infty} \int_a^b f(t)\,dt$$

If $\int_a^b f(t)\,dt$ exists for all $b \geq a$ and if the limit of this as $b\to\infty$ exists then we say the integral <u>converges</u> to the limiting value – otherwise it diverges.

<u>Ex</u>

$$\int_1^\infty \frac{1}{t^2}\,dt = \lim_{b\to\infty} \int_1^b \frac{1}{t^2}\,dt$$

$$\int_1^b t^{-2}\,dt = \left.\frac{t^{-1}}{-1}\right|_1^b = -\frac{1}{b} + \frac{1}{1} = 1 - \frac{1}{b}$$

$$\lim_{b\to\infty} 1 - \frac{1}{b} = 1 \qquad \text{so} \qquad \int_1^\infty \frac{1}{t^2}\,dt = 1$$

<u>Ex</u>

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{b\to\infty} \int_1^b t^{-p}\,dt \qquad p \neq 1$$

$$\int_1^b t^{-p}\,dt = \left.\frac{t^{-p+1}}{-p+1}\right|_1^b = \frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p} = \frac{b^{1-p}-1}{1-p}$$

$$\lim_{b\to\infty} \frac{b^{1-p}-1}{1-p} = \begin{cases} \text{d.n.e} & \text{if } p<1 \\ \frac{1}{p-1} & \text{if } p>1 \end{cases}$$

if $p=1$ then $\int_1^b \frac{1}{t}\,dt = \ln t \Big|_1^b = \ln b - \ln 1 = \ln b$

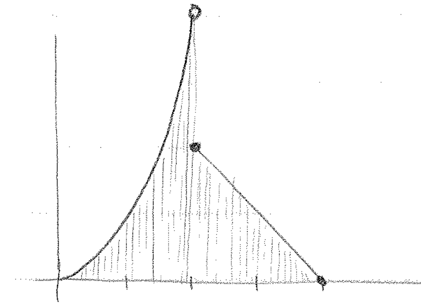
$$\lim_{b\to\infty} \ln b \Rightarrow \infty \quad \text{so} \quad \lim \text{ d.n.e}$$

$$\boxed{\int_1^\infty \frac{1}{t^p}\,dp = \frac{1}{p-1} \qquad \text{if } p>1}$$

Now, lets integrate (if possible)

open ended interval

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 4-t & 2 \le t \le 4 \end{cases}$$



$$\int_0^4 f(t)\,dt = \lim_{b\to 2^-} \int_0^b f(t)\,dt + \int_2^4 f(t)\,dt$$

$$= \lim_{b\to 2^-} \int_0^b t^2\,dt + \int_2^4 (4-t)\,dt$$

$$= \left[\lim_{b\to 2^-} \frac{b^3}{3}\right] + \left[4t - \frac{t^2}{2}\right]\Big|_2^4 = \frac{8}{3} + 2 = \frac{14}{3}$$

This is an example of a <u>piecewise continuous function</u> on $[\alpha, \beta]$

- continuous between α and β except at a finite number of points
- discontinuities are restricted to jump discontinuities, i.e. $f \to$ finite limit as $x \to$ endpoint from interior of interval.

[we are now starting to face the fact that many interesting problems have discontinuous functions related to them].

Review Thm 6.1.1 and related topics from calculus.

"comparison test"

Transform methods used in solution of differential equations

- hard problem $\rightarrow$ easier problem
- Unknown problem $\rightarrow$ known problem
- One "space" $\longrightarrow$ second "space"

We define the Laplace transform of $f(t)$ to be

$$\mathcal{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\, dt$$

We are using the integral to transform from functions $f(t)$ to functions in a different space containing functions $F(s)$ of s.

Thm 6.1.2

If ① f is piecewise continuous on $0 \le t \le A \quad \forall A > 0$

② $|f(t)| \le K e^{at}$ when $t \ge M$ ($K>0$, $M>0$, a = any real const)
(i.e. $f(t)$ is bounded by an exponential function)

Then

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists $\forall\ s > a$.

Proof: $$F(s) = \underbrace{\int_0^M e^{-st} f(t)\, dt}_{} + \int_M^\infty e^{-st} f(t)\, dt$$

Since $e^{-st} f(t)$ is piecewise cont. this integral exists

$$\left| \int_M^\infty e^{-st} f(t) dt \right| \le \int_M^\infty e^{-st} |f(t)| dt \le \int_M^\infty e^{-st} K e^{at} dt = K \int_M^\infty e^{(a-s)t} dt = \lim_{b\to\infty} K \frac{e^{(a-s)t}}{a-s}\Bigg|_M^b = -\frac{e^{(a-s)M}}{a-s} \cdot K$$

triangle inequality

if $a < s$.

Ex Compute $\mathcal{L}\{f(t)\}$ if $f(t)=1$

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^b = \boxed{\frac{1}{s}} \quad ; \; s>0$$

Ex Compute $\mathcal{L}\{f(t)\}$ if $f(t)=t$

$$F(s) = \lim_{b\to\infty}\int_0^b e^{-st}t\,dt$$

$$\int_0^b e^{-st}t\,dt = -\frac{t\,e^{-st}}{s}\Big|_0^b + \int_0^b \frac{e^{-st}}{s}\,dt = -\frac{be^{-sb}}{s} - \frac{e^{-st}}{s^2}\Big|_0^b$$

$u=t \quad dv=e^{-st}dt$

$du=dt \quad v=-\frac{e^{-st}}{s}$

$$= -\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2}$$

$$\lim_{b\to\infty}\left[\frac{1}{s^2} - \frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2}\right] = \boxed{\frac{1}{s^2}} \quad ; \; s>0$$

Ex $f(t) = \cos at$

$$F(s) = \lim_{b\to\infty}\int_0^b e^{-st}\cos at\,dt$$

$$I = \int_0^b e^{-st}\cos at\,dt = e^{-st}\frac{\sin at}{a}\Big|_0^b + \int_0^b \frac{s}{a}e^{-st}\sin at\,dt$$

$u=e^{-st} \quad dv=\cos at\,dt$

$du=-se^{-st}dt \quad v=\frac{\sin at}{a}$

$u=e^{-st} \quad dv=\sin at$

$du=-se^{-st} \quad v=-\frac{\cos at}{a}$

$$I = e^{-sb}\frac{\sin ab}{a} + \frac{s}{a}\left[-e^{-st}\frac{\cos at}{a}\Big|_0^b - \int_0^b \frac{s}{a}e^{-st}\cos at\,dt\right]$$

$$= e^{-sb}\frac{\sin ab}{a} - \frac{s}{a^2}e^{-sb}\cos ab + \frac{s}{a^2} - \frac{s^2}{a^2}I$$

$$\left(1+\frac{s^2}{a^2}\right)I = \frac{1}{a}e^{-sb}\sin ab - \frac{s}{a^2}e^{-sb}\cos ab + \frac{s}{a^2}$$

If $s>0$ then $\lim b\to\infty$ gives

$$\left(1+\frac{s^2}{a^2}\right)F(s) = \frac{s}{a^2} \quad \to \quad \boxed{F(s) = \frac{s}{a^2+s^2} \; ; \; s>0}$$

Finally, we show that $\mathcal{L}\{\ \}$ is a linear operator

$$\mathcal{L}\{cf(t)+dg(t)\} = \int_0^\infty e^{-st}[cf(t)+dg(t)]\,dt$$

$$= \int_0^\infty e^{-st}cf(t)\,dt + \int_0^\infty e^{-st}dg(t)\,dt$$

$$= c\int_0^\infty e^{-st}f(t)\,dt + d\int_0^\infty e^{-st}g(t)\,dt$$

$$= c\mathcal{L}\{f(t)\} + d\mathcal{L}\{g(t)\}$$

This is true (of course) since the integral is a linear operator.