5.6 Series Solution near a Regular Singular Point I

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \qquad P(0)=0$$

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad p(x) = \frac{Q(x)}{P(x)} \qquad q(x) = \frac{R(x)}{P(x)}$$

Notice how this resembles Euler's Eq. ⇒

$$x^2 y'' + x\,[x p(x)]\,y' + [x^2 q(x)]\,y = 0$$

These terms will be analytic since $x=0$ is a regular sing. pt.

$x p(x)$ and $x^2 q(x)$ must be expressed in terms of a series expansion about $x=0$

Assume a solution $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots)$
$= x^r \sum_{n=0}^{\infty} a_n x^n$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-2}$$

METHOD OF FROBENIUS

5.6 Series Solutions near a Regular Singular Point I

Ex 1. $2xy'' + y' + xy = 0$

in the form $y'' + \frac{1}{2x}y' + \frac{1}{2}y = 0$ we can see that $x=0$ is a singular point. Observe that

$$\lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2} \qquad \lim_{x\to 0} x^2\frac{1}{2} = 0$$

So $x=0$ is a regular singular point. Now, we could assume a solution of $y = x^r$, but in general (unlike Euler equations) we will need series solutions since the coefficient function are in general not constant or simple as in Euler eq.

We use the form $x^2y'' + \frac{x}{2}y' + \frac{x^2}{2}y = 0$

Essentially e want it to look like the Euler eq as much as possible since it is our canonical example of a d.e. with a regular singular point at zero.

Assume $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

So the d.e. becomes

$$x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + \frac{x}{2}\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \frac{x^2}{2}\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} \frac{a_n}{2}(n+r) x^{n+r} + \sum_{n=0}^{\infty} \frac{a_n}{2} x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+r) \left[(n+r-1)+\tfrac{1}{2}\right] x^{n+r} + \sum_{n=2}^{\infty} \frac{a_{n-2}}{2} x^{n+r} = 0$$

So

$$a_0 r(r-\tfrac{1}{2}) x^r + a_1 (1+r)(r+\tfrac{1}{2}) x^{1+r} + \sum_{n=2} \left[a_n(n+r)(n+r-\tfrac{1}{2}) + \frac{a_{n-2}}{2}\right] x^{n+r} = 0$$

Since all coefficients of like powers of x must match on both sides of the "=" sign and we require that $a_0 \neq 0$ so

$$r(r-\tfrac{1}{2}) = 0 \qquad \left[\begin{array}{l} a_0 \neq 0 \text{ allows us to fix the} \\ \text{value for } r \end{array}\right]$$

This eq. is called the <u>indicial equation</u> for the diff. eq. and indicates what values r should take on. So

$$r_1 = \tfrac{1}{2}, \quad r_2 = 0 \qquad [\text{exponents of singularity}]$$

Notice that $a_1(1+r)(r+\tfrac{1}{2}) = 0$ is only true if $a_1 = 0$. From the Sum, we see that

$$a_n(n+r)(n+r-\tfrac{1}{2}) + \frac{a_{n-2}}{2} = 0 \quad \text{so}$$

$$a_n = \frac{-a_{n-2}}{2(n+r)(n+r-\frac{1}{2})} = -\frac{a_{n-2}}{(n+r)[2(n+r)-1]}$$

$$r_1 = \tfrac{1}{2}: \quad y_1 = a_0 x^{\frac{1}{2}} \left[1 - \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 5 \cdot 4 \cdot 9} - \frac{x^6}{2 \cdot 5 \cdot 4 \cdot 9 \cdot 6 \cdot 13} + \cdots \right]$$

$$r_2 = 0: \quad y_2 = a_0 \left[1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 7} - \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 7 \cdot 6 \cdot 11} + \cdots \right]$$

We will always be able to find 1 of the 2 soln. in the fund. set like this. If $r_1 \neq r_2 + N$ for integer N, then 2 solutions can be found in this way.

Ex 6 $x^2y'' + xy' + (x-2)y = 0$

$x=0$ is a reg. sing pt.

Assume $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$x^2 \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + (x-2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[a_n(n+r)(n+r-1) + a_n(n+r) - 2a_n\right]x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

$$\left[a_0 r(r-1) + a_0 r - 2a_0\right]x^r + \sum_{n=1}^{\infty}\left[a_n\left[(n+r)(n+r-1)+(n+r)-2\right] + a_{n-1}\right]x^{n+r} = 0$$

indicial eq. $r(r-1) + r - 2 = 0 \rightarrow r^2 - 2 = 0 \rightarrow r = \pm\sqrt{2}$

$r_1 = \sqrt{2}$, $r_2 = -\sqrt{2}$

recurrence relation: $a_n\left[(n+r)^2 - 2\right] + a_{n-1} = 0$

So $a_n = \dfrac{a_{n-1}}{2-(n+r)^2}$

$r_1 = \sqrt{2}$ $\qquad a_n = \dfrac{a_{n-1}}{2 - (n+\sqrt{2})^2} = -\dfrac{a_{n-1}}{n(n+2\sqrt{2})}$

$$y_1 = a_0 x^{\sqrt{2}} \left[1 - \frac{x}{1+2\sqrt{2}} + \frac{x^2}{2!(1+2\sqrt{2})(2+2\sqrt{2})} - \frac{x^3}{3!(1+2\sqrt{2})(2+2\sqrt{2})(3+2\sqrt{2})} + \cdots \right]$$

$r_1 = -\sqrt{2}$ $\qquad a_n = -\dfrac{a_{n-1}}{n(n-2\sqrt{2})}$

$$y_2 = a_0 x^{-\sqrt{2}} \left[1 - \frac{x}{1-2\sqrt{2}} + \frac{x^2}{2!(1-2\sqrt{2})(2+2\sqrt{2})} - \frac{x^3}{3!(1-2\sqrt{2})(2-2\sqrt{2})(3-2\sqrt{2})} + \cdots \right]$$

Ex $\quad x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0 \qquad x=0$ is a reg. sing. pt by inspection

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$x^2y'' + xy' + (x^2 - \tfrac{1}{9})y = x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} - \frac{1}{9}\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} \frac{a_n}{9} x^{n+r} = 0$$

$$= \sum_{n=0}^{\infty} \left[a_n (n+r)(n+r-1) + a_n (n+r) - \frac{a_n}{9} \right] x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$a_0 \left[r(r-1) + (r) - \tfrac{1}{9} \right] x^r + a_1 \left[(1+r)(r) + (1+r) - \tfrac{1}{9} \right] x^{1+r} + \sum_{n=2}^{\infty} \left\{ a_n \left[(n+r)^2 - \tfrac{1}{9} \right] + a_{n-2} \right\} x^{n+r} = 0$$

Since $a_0 \neq 0$ we have $\underbrace{r(r-1) + r - \frac{1}{9}}_{\text{This is the indicial equation}} = r^2 - \frac{1}{9} = 0 \rightarrow r = \pm \frac{1}{3}$

$r_1 = \frac{1}{3}, \quad r_2 = -\frac{1}{3}$

$a_1 \left[(1+r)^2 - \frac{1}{9} \right] = 0 \qquad$ since $(1+r)^2 - \frac{1}{9} \neq 0$ for $r = \pm\frac{1}{3}$ we conclude that $\boxed{a_1 = 0}$

$$a_n \left[(n+r)^2 - \tfrac{1}{9} \right] + a_{n-2} = 0 \implies a_n = \frac{-a_{n-2}}{(n+r)^2 - \frac{1}{9}}$$

When $r = \frac{1}{3}$: $\quad a_n = \dfrac{-a_{n-2}}{(n+\frac{1}{3})^2 - \frac{1}{9}} = \dfrac{-a_{n-2}}{n^2 + \frac{2}{3}n} = \dfrac{-a_{n-2}}{n(n+\frac{2}{3})}$

When $r = -\frac{1}{3}$ $\quad a_n = \dfrac{-a_{n-2}}{(n-\frac{1}{3})^2 - \frac{1}{9}} = \dfrac{-a_{n-2}}{n^2 - \frac{2}{3}n} = \dfrac{-a_{n-2}}{n(n-\frac{2}{3})}$

5.6

$r = \frac{1}{3}$

a_0

$a_1 = 0$

$a_2 = \dfrac{-a_0}{2(2+\frac{2}{3})}$

$a_3 = 0$

$a_4 = \dfrac{-a_2}{4(4+\frac{2}{3})} = \dfrac{a_0}{2\cdot 4\cdot(2+\frac{2}{3})(4+\frac{2}{3})}$

$$y_1 = x^{1/3}\left[1 - \frac{1}{4(1+\frac{1}{3})}x^2 + \frac{1}{32(1+\frac{1}{3})(2+\frac{1}{3})}x^4 - \frac{1}{192(1+\frac{1}{3})(2+\frac{1}{3})(3+\frac{1}{3})}x^6 + \cdots\right]$$

$r = -\frac{1}{3}$

a_0

$a_1 = 0$

$a_2 = \dfrac{-a_0}{2(2-\frac{2}{3})}$

$a_3 = 0$

$a_4 = \dfrac{-a_2}{4(4-\frac{2}{3})} = \dfrac{a_0}{2\cdot 4(2-\frac{2}{3})(4-\frac{2}{3})}$

$$y_2 = x^{-1/3}\left[1 - \frac{1}{4(1-\frac{1}{3})}x^2 + \frac{1}{32(1-\frac{1}{3})(2-\frac{1}{3})}x^4 - \frac{1}{192(1-\frac{1}{3})(2-\frac{1}{3})(3-\frac{1}{3})}x^6 + \cdots\right]$$