5.4 Regular Singular Points

Consider $\quad xy'' + y' = 0$

If we put this into the form $y'' + p(x)y' + q(x)y = 0$ we get

$$y'' + \frac{1}{x}y' = 0$$

We cannot use our existance & uniqueness thm. on an interval which contains zero.

To solve, take $v(x) = y'$ so the eq. becomes

$$xv' + v = 0$$

$$(xv)' = 0 \quad \rightarrow \quad xv = c_1$$

$$v = \frac{c_1}{x} \quad \rightarrow \quad y = c_1 \ln|x| + c_2$$

So for $x \neq 0$ we have $y = c_1 \ln|x| + c_2$ with the two lin. indep. solutions $\ln|x|$ and 1.

Notice that one of these solutions is bounded as $x \to 0$ while the other goes to $-\infty$.

We call $x = 0$ (in this problem) a <u>singular</u> point.

For the general 2^{nd} order eq: $\quad P(x)y'' + Q(x)y' + R(x)y = 0$
We will need to find a series solution. Unfortunately, we can't expand about the point x_0 if this is a singular point.

Ex Identify the singular points in the following differential equations:

A) $(1-x^2)\,y'' - 2xy' + \alpha(\alpha+1)\,y = 0$ (Legendre Eq)

Sing. points occur when $P(x) = (1-x^2) = 0$

$x = \pm 1$ are the singular points.

B) $(\sin x)\,y'' + (\cos x)\,y = 0$

$\sin x = 0$ if $\underbrace{x = n\pi\,, \quad n = 0, \pm 1, \pm 2, \ldots}_{\text{sing. points}}$

c) $x^2 y'' + xy' + (x^2 - \nu^2)\,y = 0$ (Bessel eq)

Sing. point is $x = 0$

Next, we describe two different types of singular points.

- difference between the two relates to how "bad" the singularity is — i.e., how difficult is the problem to solve.

First, write the eq. in the form $y'' + p(x)\,y' + q(x)\,y = 0$

If this eq. has a singular point x_0 [either $p(x)$ or $q(x)$ (or both) is non-analytic at x_0] then

- x_0 is a <u>regular singular point</u> if

$$\lim_{x \to x_0} (x - x_0)\, p(x) \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 q(x)$$

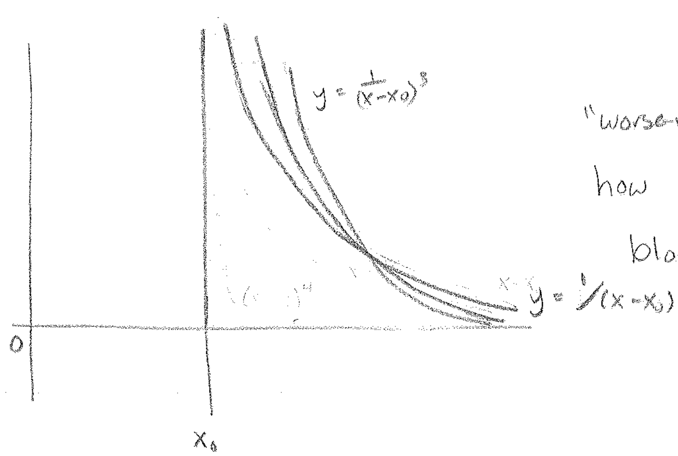
$$p(x) = \frac{Q(x)}{P(x)} \qquad q(x) = \frac{R(x)}{P(x)}$$

are both finite. (or $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ are both analytic).

- Otherwise, x_0 is an <u>irregular singular point</u>.

If both $p(x)$ and $q(x)$ are rational functions, then we just need $(x-x_0)\, p(x)$ and $(x-x_0)^2 q(x)$ to be finite as $x \to x_0$

This means $p(x)$ is "no worse" than $(x-x_0)^{-1}$ and $q(x)$ is "no worse" than $(x-x_0)^{-2}$.



"worseness" has to do with how quickly the function blows up as $x \to x_0$.

Ex Find the singular points of

$$x^2(1-x^2)y'' + \frac{2}{x}y' + 4y = 0$$

and label each as regular or irregular.

Rewrite as $x^3(1-x^2)y'' + 2y' + 4xy = 0$

So $P(x)$, $Q(x)$ and $R(x)$ are all polynomials (or analytic). Then singular points are where $P(x) = 0$

$$x^3(1-x^2) = 0 \quad \text{at} \quad x = 0, \pm 1$$

$x=0$: $\quad (x-0)\dfrac{2}{x^3(1-x^2)}$ is not finite as $x \to 0$

$x=+1$: $\quad (x-1)\dfrac{2}{x^3(1-x^2)} = \dfrac{-2}{x^3(1+x)}$ is finite as $x \to 1$

$\quad (x-1)^2\dfrac{4x}{x^3(1-x^2)} = \dfrac{-4(x-1)}{x^2(1+x)}$ is finite as $x \to 1$

$x=-1$: $\quad (x+1)\dfrac{2}{x^3(1-x^2)} = \dfrac{2}{x^3(1-x)}$ is finite as $x \to -1$

$\quad (x+1)^2\dfrac{4x}{x^3(1-x^2)} = \dfrac{4(x+1)}{x^2(1-x)}$ is finite as $x \to -1$

$\therefore$ $x=0$ is an irregular singular point

$x = \pm 1$ are regular singular points

5.5 Euler Equations

The Euler Eq is $L[y] = x^2 y'' + \alpha x y' + \beta y = 0$

which has 0 as a regular singular point. The eq.

$$(x - x_0)^2 y'' + \alpha(x - x_0) y' + \beta y = 0$$

is the Euler eq with x_0 as a reg. singular point

Now, consider $L[y] = x^2 y'' + \alpha x y' + \beta y = 0$ and let us restrict ourselves to $x > 0$. We can assume a solution of $y = x^r$.

<u>Ex</u> $x^2 y'' + 4xy' + 2y = 0$ assume $y = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1) x^{r-2}$

$$x^2 r(r-1) x^{r-2} + 4x r x^{r-1} + 2x^r = 0$$

$$(r(r-1) + 4r + 2) x^r = 0$$

So $r(r-1) + 4r + 2 = 0 \rightarrow r^2 + 3r + 2 = (r+2)(r+1) = 0$

so $r_1 = -1$, $r_2 = -2$ are 2 distinct roots.

$$y_1 = x^{-1}, \quad y_2 = x^{-2}$$

The wronskian of y_1 & y_2 $W(y_1, y_2) = \begin{vmatrix} \frac{1}{x} & \frac{1}{x^2} \\ -\frac{1}{x^2} & -\frac{2}{x^3} \end{vmatrix} = -\frac{2}{x^4} + \frac{1}{x^4} = -\frac{1}{x^4}$

$-\frac{1}{x^4} \neq 0$

Which tells us that y_1 and y_2 form a fundamental set of solutions. Thus, the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 x^{-1} + c_2 x^{-2} \quad \text{when} \quad x \neq 0$$

☆ The key was that in this example the two values of r we found were real and distinct.

Now consider $x^2 y'' - 3xy' + 4y = 0$. Assume $y = x^r$ then

$$y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2} \quad \text{and}$$

$$L[x^r] = x^2 r(r-1)x^{r-2} - 3x\, r x^{r-1} + 4x^r = 0$$

$$(r(r-1) - 3r + 4)x^r = 0 \Rightarrow r(r-1) - 3r + 4 = 0$$

So

$$r^2 - 4r + 4 = 0 \rightarrow (r-2)(r-2) = 0$$

$$r_1 = 2, \quad r_2 = 2 \quad \text{so we have real repeated roots.}$$

So $y_1 = x^2$ but we need to find another linearly independent solution.

Start with $L[x^r] = (r-2)^2 x^r$. We can differentiate wrt r. We assume $x > 0$ and note that $\frac{\partial}{\partial r} x^r = x^r \ln x$. So

$$\frac{\partial}{\partial r} L[x^r] = L\left[\frac{\partial}{\partial r} x^r\right] = L[x^r \ln x] = \frac{\partial}{\partial r}(r-2)^2 x^r = 2(r-2)x^r + (r-2)^2 x^r \ln x$$

but $2(r-2)x^r + (r-2)^2 x^r \ln x = 0$ when $r = 2$.

Thus $L[x^r \ln x] = 0$ at $r=2$ so $x^2 \ln x$ is also a solution.

are x^2 and $x^2 \ln x$ linearly independent?

$$W(x^2, x^2\ln x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix} = (2x^3 \ln x + x^3) - 2x^3 \ln x$$

$$= x^3 \neq 0 \quad \text{if } x \neq 0$$

our sing. point.

$\therefore \quad y_2 = x^2 \ln x$

So

$$\left.\begin{aligned} y = c_1 y_1 + c_2 y_2 &= c_1 x^2 + c_2 x^2 \ln x \\ &= (c_1 + c_2 \ln x)\, x^2 \end{aligned}\right\} \text{ valid for } x>0$$

What about if $x<0$? Let $-\xi = x$ and set $u(\xi) = y(x)$ so that

$$y' = \frac{d}{dx} u(\xi) = \frac{du}{d\xi}\frac{d\xi}{dx} = -u' \quad \text{if} \quad u' = \frac{du}{d\xi}$$

$$y'' = \frac{d}{dx} y' = \frac{d(-u')}{dx} = u'' \quad \text{if} \quad u'' = \frac{d^2u}{d\xi^2}$$

So we get $\quad (-\xi)^2 u'' - 3(-\xi)(-u') + 4u = 0$

or $\quad \xi^2 u'' - 3\xi u' + 4u = 0$

which has the solution

$$u(\xi) = (c_1 + c_2 \ln \xi)\, \xi^2, \quad \xi > 0, \quad x<0$$

So in general

$$y = (c_1 + c_2 \ln|x|)\,|x|^2$$

Done Real and distinct r, real and repeated r, need to handle complex conjugates.

Ex $\quad L[y] = x^2y'' + 3xy' + 2y = 0$

$$y = x^r \rightarrow \quad [r(r-1) + 3r + 2]x^r = 0$$

so $\quad r(r-1) + 3r + 2 = r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i$$

So $y_1 = x^{-1+i}$, $y_2 = x^{-1-i}$ Well, what does this mean?

Use $x^r = e^{r\ln x}$ so if $r = \lambda \pm i\mu$ we get

$$x^r = x^{\lambda \pm i\mu} = e^{(\lambda \pm i\mu)\ln x} = e^{\lambda \ln x} e^{\pm i\mu \ln x}$$
$$= x^{\lambda}[\cos(\mu \ln x) \pm i \sin(\mu \ln x)]$$

Now, just as when we dealt with complex roots of the characteristic equation, we can replace these solutions with strictly real-valued functions

$$x^{\lambda}\cos(\mu \ln x) \quad \text{and} \quad x^{\lambda}\sin(\mu \ln x)$$

So we can determine the solution to be

$$y = c_1 x^{-1}\cos(\ln x) + c_2 x^{-1}\sin(\ln x).$$