5.2 Series Solutions near an Ordinary Point I

Why series?

- needed for non-constant coefficients
- get solutions as accurate as we desire

<u>Ex</u> $y''-xy'-y=0$ We assume a solution centered on $x_0=0$.

assume

$$y=\sum_{n=0}^{\infty} a_n x^n$$

To identify a_n's, substitute this back into o.d.e.

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

So $$y''-xy'-y=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-x\sum_{n=1}^{\infty} n a_n x^{n-1}-\sum_{n=0}^{\infty} a_n x^n=0$$

We now want to group parts of the sums by powers of x. By changing the index of summation, we can express all powers of x as x^n

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty} n a_n x^n-\sum_{n=0}^{\infty} a_n x^n=0$$

can change lower index to zero since then the term is zero.

$$\sum_{n=0}^{\infty}\Big[\underbrace{(n+2)(n+1)a_{n+2}-na_n-a_n}\Big]x^n=0$$

To be true $\forall x$, this must be zero.

$$(n+2)(n+1)a_{n+2}-(n+1)a_n=0$$ <u>Recurrence Relation</u>

So $(n+2)(n+1)a_{n+2} = (n+1)a_n$

$(n+2)a_{n+2} = a_n$

Recurrence relation "skips" one term. Recall that we need 2 arb. constants

$a_0 = a_0$

$2a_2 = a_0 \rightarrow a_2 = a_0/2$

$4a_4 = a_2 \rightarrow a_4 = a_0/2\cdot4$

$6a_6 = a_4 \rightarrow a_6 = a_0/2\cdot4\cdot6$

$8a_8 = a_6 \rightarrow a_8 = a_0/2\cdot4\cdot6\cdot8$

$\vdots$

$a_1 = a_1$

$3a_3 = a_1 \rightarrow a_3 = a_1/3$

$5a_5 = a_3 \rightarrow a_5 = a_1/3\cdot5$

$7a_7 = a_5 \rightarrow a_7 = a_1/3\cdot5\cdot7$

$9a_9 = a_7 \rightarrow a_9 = a_1/3\cdot5\cdot7\cdot9$

So $$y(x) = a_0\left[1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \frac{x^8}{384} + \cdots\right]$$

$$+ a_1\left[x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \frac{x^9}{945} + \cdots\right]$$

$$= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n n!\, x^{2n+1}}{(2n+1)!}$$

since $$3\cdot5\cdot\cdots\cdot(2n+1) = \frac{(2n+1)!}{2\cdot4\cdots\cdot2n} = \frac{(2n+1)!}{2^n n!}$$

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (1)$$

is the general homogeneous, Linear 2nd order equation.

If $P(x) \neq 0$ for x in some neighborhood around x_0, then we can divide by $P(x)$ and the existence & uniqueness Thm guarentees that there will be a unique solution of

$$y'' + p(x)y' + q(x) = 0 \qquad (2)$$

In this case, x_0 is an <u>ordinary point</u>. If $P(x_0) = 0$, then x_0 is a <u>singular point</u>.

So assuming x_0 is an ordinary point of (2), we assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Also - need to put D.E into form $y'' + \bar{p}(x-x_0)y' + \bar{q}(x-x_0) = 0$

<u>Ex</u> Again, consider $y'' - xy' - y = 0$, but now assume the solution is centered about $x_0 = 1$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad y' = \sum_{n=1}^{\infty} n\, a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=1}^{\infty} n\, a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \left[(x-1) + 1\right] \sum_{n=1}^{\infty} n\, a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

need to express coef. in terms of $(x-1)$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n\, a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} - a_n \right] (x-1)^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - (n+1) a_n = 0$$

$$(n+2) a_{n+2} - a_{n+1} - a_n = 0$$

$$a_{n+2} = \frac{a_{n+1} + a_n}{n+2}$$

n	a_n
0	–
1	–
2	$\frac{a_1+a_0}{2}$
3	$\frac{a_2+a_1}{3} = \frac{1}{3}\left[\frac{a_1+a_0}{2}+a_1\right] = \frac{3a_1+a_0}{6}$
4	$\frac{a_3+a_2}{4} = \frac{1}{4}\left[\frac{3a_1+a_0}{6}+\frac{a_1+a_0}{2}\right] = \frac{1}{4}\left[\frac{6a_1+4a_0}{6}\right] = \frac{3a_1+2a_0}{12}$
5	$\frac{a_4+a_3}{5} = \frac{1}{5}\left[\frac{3a_1+2a_0}{12}+\frac{3a_1+a_0}{6}\right] = \frac{1}{5}\left[\frac{9a_1+4a_0}{12}\right] = \frac{9a_1+4a_0}{60}$
$\vdots$	

So
$$y = a_0\left[1+\tfrac{1}{2}(x-1)^2+\tfrac{1}{6}(x-1)^3+\tfrac{1}{6}(x-1)^4+\tfrac{1}{15}(x-1)^5+\cdots\right]$$
$$+\, a_1\left[(x-1)+\tfrac{1}{2}(x-1)^2+\tfrac{1}{2}(x-1)^3+\tfrac{1}{4}(x-1)^4+\tfrac{3}{20}(x-1)^5+\cdots\right]$$

[Note: P, Q, R or p & q must be expressed as polynomials or series in terms of $x - x_0$.]