5.1 Review of Power Series

A <u>power series</u> has the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ where x_0 is a fixed point.

The power series <u>converges at x</u> if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n (x-x_0)^n$$

exists.

The power series <u>converges absolutely at x</u> if the series

$$\sum_{n=0}^{\infty} |a_n (x-x_0)^n|$$

converges.

If a power series converges absolutely at x then it converges at x. The converse is not true in general.

The two most frequently used tests for convergence are the <u>n^{th} root test</u> and the <u>ratio test</u>.

<u>n^{th} root test:</u> the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely at x if

$$\lim_{n\to\infty} \sqrt[n]{|a_n (x-x_0)^n|} = L < 1$$

This reduces to

$$|x-x_0| \lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$$

<u>ratio test</u>: The series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely at x if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = L < 1$$

or

$$|x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$$

(note: this test only applies if all $a_n \neq 0$, $n > N$ for some N)

In each case L is a constant. If $L<1$ then we have abs. convergence. If $L>1$ we know that the series <u>diverges</u>. If $L=1$ the test is inconclusive.

For any power series there is a nonnegative number ρ (rho) called the <u>radius of convergence</u> such that if

$$|x-x_0| < \rho$$

then the series converges absolutely and if

$$|x-x_0| > \rho$$

the series diverges. It may either converge or diverge at

$$|x-x_0| = \rho.$$

<u>Ex</u> Determine the radius of convergence of

a) $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$ b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

5.1

a) $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$ — in this case $x_0 = 0$

We will use the ratio test
(n^{th} root test would also work)

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}} x^{n+1}}{\frac{n}{2^n} x^n} \right|$$

(note: in this example $x_0 = 0$)

$$= |x| \lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{1}{2} \cdot \frac{n+1}{n} \right| = \frac{1}{2} |x| \lim_{n\to\infty} \left| \frac{n+1}{n} \right|$$

$$= \frac{1}{2} |x|$$

We want this to be less than one, so

$$\frac{1}{2}|x| < 1$$

$$|x| < 2 \qquad \boxed{\rho = 2}$$

The <u>interval of convergence</u> is $x_0 - \rho < x < x_0 + \rho$
but may also include either endpoint.

$x_0 = 0$

Check $x = -2$: $\sum_{n=0}^{\infty} \frac{n}{2^n} (-2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot n$

<u>diverges</u> since terms don't go to 0

Check $x = +2$: $\sum_{n=0}^{\infty} \frac{n}{2^n} (2)^n = \sum_{n=0}^{\infty} n$

diverges since terms don't go to 0

$\boxed{\text{interval of convergence: } -2 < x < 2}$

b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| x^2 \frac{n!}{(n+1)\,n!} \right|$$

$$= x^2 \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0$$

Since this limit is zero for all x, the radius of convergence is infinite: $\boxed{\rho = \infty}$

Now, assume

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

(note: n is a "dummy parameter", much like the variable of integration in a definite integral)

In this case we say the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ <u>converges to $f(x)$</u>. There may be restrictions on the interval – we'll assume

$$|x-x_0| < \rho$$
$$\rho > 0$$

① $$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$$

(Convergent) Series may be added term-by-term.

② $f(x)\,g(x) = \left[\sum_{n=0}^{\infty} a_n (x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n (x-x_0)^n\right]$

$$= \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

division is also defined whenever $g(x) \neq 0$.

③ f and all its derivatives exist on $|x-x_0| < \rho$

and
$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

The Taylor series for a function $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$ [f is analytic at x_0]

④ If $\sum a_n (x-x_0)^n = \sum b_n (x-x_0)^n$ for each x

then $a_n = b_n$ for $n = 0, 1, 2, \ldots$

It is also often useful to "shift" the index of summation [remember that it's just a dummy parameter].

$$\sum_{n=2}^{\infty} a_n (x-x_0)^n = a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \cdots$$

But...
$$\sum_{n=0}^{\infty} a_{n+2} (x-x_0)^{n+2} = a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \cdots$$

We can always accomplish this by

1. Replacing n (or whatever the index of summation happens to be) by $n \pm a$ where a is the amount we want to shift by.
   VIN: (very important note) always use parentheses around the n that's replaced.

Ex

Change $\sum_{n=1}^{\infty} a_n (x-x_0)^{2n}$ so that the index of summation begins at zero.

$$\sum_{n+1=1}^{\infty} a_{(n+1)} (x-x_0)^{2(n+1)}$$

$$\sum_{n=0}^{\infty} a_{n+1} (x-x_0)^{2n+2}$$ done.