4.2 Homogeneous Equations with constant Coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \qquad a_i \text{ real}$$

Assume a solution of the form $y = e^{rx}$

$$L[e^{rx}] = a_0 r^n e^{rx} + a_1 r^{n-1} e^{rx} + \cdots + a_n e^{rx} = 0$$
$$= e^{rx}\left[a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n\right] = 0$$

So

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$ is the characteristic eq.

We need to find the roots of this polynomial. The roots will be

- real & distinct
- real & repeated
- complex conjugates (distinct or repeated)

or combinations of these. We expect to find n roots, counting multiplicities.

Once the roots $r_1, \cdots, r_n$ are determined, we then can proceed with the construction of our solution.

If we arrange the roots so that the first m, $r_1, \cdots, r_m$ are real and distinct, then

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_m e^{r_m x}$$

is a solution, and if $m = n$, this is the <u>general solution</u> of $L[y] = 0$.

It is important, as in the $n=2$ case, that $e^{r_i x}$ $i=1,\dots,m$ are linearly independent. Since $r_1,\dots,r_m$ are distinct we have the independence we need. See 4.2 #40 for proof.

Suppose that $r_{m+1},\dots,r_{m+s}$ are a single root with multiplicity s. Then if all these roots are called α

$$y = C_{m+1}e^{\alpha x} + C_{m+2}\,x\,e^{\alpha x} + \dots + C_{m+s}x^{s-1}e^{\alpha x}$$

is also a solution of $L[y]=0$. To prove this we need only show that $x^p e^{\alpha x}$, $p<s$, is a solution since $L[y]$ is a linear operator.

Proof sketched in problem 41 of section 4.2.

$$L[e^{rx}] = e^{rx}(r-\alpha)^s \underbrace{H(r)}_{\text{all other factors of characteristic polynomial.}}$$

Now diff wrt r on both sides, interchanging diff wrt r and x.

$$L[xe^{rx}] = xe^{rx}(r-\alpha)^s H(r) + e^{rx}s(r-\alpha)^{s-1}H(r) + e^{rx}(r-\alpha)^s H'(r)$$

if $r=\alpha$

$$L[xe^{\alpha x}] = e^{\alpha x}\left[x\cancelto{0}{(\alpha-\alpha)^s}H(\alpha) + s\cancelto{0}{(\alpha-\alpha)^{s-1}}H(\alpha) + \cancelto{0}{(\alpha-\alpha)^s}H'(\alpha)\right]$$
$$=0 \quad \text{if} \quad s\geq 2$$

diff again to get $L[x^2e^{rx}]$ and sub. $r=\alpha$ to find
$$L[x^2e^{\alpha x}]=0 \quad \text{if} \quad s\geq 3$$

Continuing in this manner we see that $x^p e^{\alpha x}$ is a solution for any $p<s$.

Now that all real roots have been dealt with, we consider the complex roots. Since the coef. of the o.d.e are real we know that complex roots will occur in conjugate pairs.

For each distinct complex conjugate pair $r = \lambda \pm i\mu$ both

$$e^{\lambda x}\cos\mu x \qquad e^{\lambda x}\sin\mu x$$

are solution of $L[y]=0$

If any complex roots are repeated s times, then

$$\begin{array}{ll} e^{\lambda x}\cos\mu x & e^{\lambda x}\sin\mu x \\ xe^{\lambda x}\cos\mu x & xe^{\lambda x}\sin\mu x \\ \vdots & \vdots \\ x^{s-1}e^{\lambda x}\cos\mu x & x^{s-1}e^{\lambda x}\sin\mu x \end{array}$$

are also solutions of $L[y]=0$.

<u>Ex</u> $y^{iv} + 2y'' - 8y' + 5y = 0$

$$\rightarrow r^4 + 0r^3 + 2r^2 - 8r + 5 = 0$$

$$\rightarrow r = \underbrace{1, 1}, \underbrace{-1+2i, -1-2i}$$ ← derived on next page

$$e^x, \; xe^x \qquad e^{-x}\cos 2x \quad e^{-x}\sin 2x$$

general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-x}\cos\mu x + c_4 e^{-x}\sin\mu x$$

$y^{iv} + 2y'' - 8y' + 5y = 0$ assume $y = e^{rx}$

$r^4 + 2r^2 - 8r + 5 = 0$

leading coefficient is 1, factors are ± 1
constant is 5, factors are $\pm 1, \pm 5$

Possible rational roots are $\frac{\pm 1}{\pm 1}$, $\frac{\pm 5}{\pm 1}$ or $\pm 1, \pm 5$

using Rational Root Theorem

$r = 1$ is a root: $1^4 + 2 \cdot 1^2 - 8 \cdot 1 + 5 = 0$ ✓

$r = -1$ is not a root: $1 + 2 + 8 + 5 \neq 0$

$r = 5$ is not a root

$r = -5$ is not a root

Divide $(r-1)$ out of polynomial

$$
\begin{array}{r|l}
 & r^3 + r^2 + 3r - 5 \\
\hline
r-1 & r^4 \quad + 2r^2 - 8r + 5 \\
 & \underline{r^4 - r^3} \\
 & \quad r^3 + 2r^2 \\
 & \quad \underline{r^3 - r^2} \\
 & \qquad 3r^2 - 8r \\
 & \qquad \underline{3r^2 - 3r} \\
 & \qquad\quad -5r + 5 \\
 & \qquad\quad \underline{-5r + 5} \\
 & \qquad\qquad 0
\end{array}
$$

So $r^3 + r^2 + 3r - 5 = 0$.

Possible roots again are $\pm 1, \pm 5$

$r = 1$ is a root: $1 + 1 + 3 - 5 = 0$

$r = -1$ is not a root: $-1 + 1 - 3 - 5 \neq 0$

$r = \pm 5$ is not a root

$$
\begin{array}{r|l}
 & r^2 + 2r + 5 \\
\hline
r-1 & r^3 + r^2 + 3r - 5 \\
 & \underline{r^3 - r^2} \\
 & \quad 2r^2 + 3r \\
 & \quad \underline{2r^2 - 2r} \\
 & \qquad 5r - 5 \\
 & \qquad \underline{5r - 5} \\
 & \qquad\quad 0
\end{array}
$$

So $r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$

$r = 1, 1, -1-2i, -1+2i$ are our roots

Suggestions for finding roots of higher order polynomials:

① Given $a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$

If $r = p/q$ is a rational root in simplest form
then p is a factor of a_n (constant term coefficient) and
q is a factor of a_0 (leading coef)

② Once one root α is found, the polynomial can be reduced by dividing by $(\alpha - x)$

$$\frac{P_n(x)}{(\alpha - x)} \rightarrow \text{new } P_{n-1}(x)$$

③ Graphing or root-finding tools also will work