4.1 General Theorey of n^{th} Order Linear Equations

Most concepts from second orderequations generelize to n^{th} order equations

$$P_0(x)\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n(x)\,y = G(x)$$

If $P_0(x) \neq 0$ on some interval I, then

$$L[y] = y^{(n)} + p_1(x)\,y^{(n-1)} + p_2(x)\,y^{(n-2)} + \cdots + p_n(x)\,y = g(x).$$

The general solution of an n^{th} order eq. will involve n constants, which in turn will require n initial conditions to generate n eq. to find the n constants.

Thm 4.1.1

If the functions p_i $i=1,\ldots,n$ and g are continuous throughout the open interval I, then $\exists$ a unique $y=\phi(x)$ which satisfies both the equation $L[y]=g(x)$ and the initial conditions

$$y(0)=y_0,\ y'(0)=y_0',\ \ldots,\ y^{(n-1)}(0)=y_0^{(n-1)}.$$

This solution exists throughout I.

Given the homogeneous eq. $L[y]=0$, we expect to find a general solution of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

and if $y_1,\ldots,y_n$ form a **fundamental set of solutions** then every solution of $L[y]=0$ is a linear combination of $y_1,\ldots,y_n$

In this case, the wronskian $W(y_1, y_2, \dots, y_n)$ is non zero at at least one point

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If $L[y] = g(x)$ is nonhomogeneous, we will first solve the corresponding homogeneous eq and then use undetermined coef or variation of parameters to find a particular solution

Ex show that $1, \cos x, \sin x$ form a fundamental set of solutions for $y''' + y' = 0$

1: $y = 1$, $y' = 0$, $y'' = 0$, $y''' = 0$ $\quad 0 + 0 = 0$ ✓

cos x: $y = \cos x$, $y' = -\sin x$, $y'' = -\cos x$, $y''' = \sin x$ $\quad \sin x + (-\sin x) = 0$ ✓

sin x: $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, $y''' = -\cos x$ $\quad -\cos x + \cos x = 0$ ✓

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1 \cdot \left((-\sin x)(-\sin x) - (-\cos x)(\cos x) \right)$$
$$= \sin^2 x + \cos^2 x = 1 \neq 0$$

$\therefore$ $y = 1$, $y = \cos x$, $y = \sin x$ are three linearly independent solutions of the 3rd order equation $y''' + y' = 0$ therefore they form a fundamental set of solutions.

4.1

Def. The functions $f_1, f_2, \ldots, f_n$ are <u>linearly dependent</u> on the interval I, if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = 0$$

for all $t \in I$.

If the set of functions is not linearly dependent it is <u>linearly independent</u>

Ex. Determine if the set of functions

$$f_1(t) = (1+t), \quad f_2(t) = t + t^2, \quad f_3(t) = 1 - t^2$$

is linearly dependent or independent on $-\infty < t < \infty$.

$$a(1+t) + b(t+t^2) + c(1-t^2) = 0$$

$$a + at + bt + bt^2 + c - ct^2 = 0$$

$$(a+c) + (a+b)t + (b-c)t^2 = 0$$

So
$$\begin{aligned} a+c &= 0 \\ a+b &= 0 \\ b-c &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so} \quad \begin{aligned} a + c &= 0 \\ b - c &= 0 \end{aligned} \Rightarrow \begin{aligned} a &= -c \\ b &= c \end{aligned}$$

if $c = 1$ and $a = -1$, $b = 1$ then

$$-1(1+t) + 1(t+t^2) + 1(1-t^2)$$
$$= -1 - t + t + t^2 + 1 - t^2$$
$$= 0 \ \checkmark$$

So the functions are linearly dependent on $-\infty < t < \infty$