3.7 Variation of Parameters

<u>Ex</u> $y'' + 9y = 9\sec^2 3x \qquad 0 < x < \pi/6$

not a sum or product of poly., sine & cosine, exp.

und. coeff. will not work (easily)

Lets solve $y'' + 9y = 0 \;:\; \rightarrow \quad y = c_1 \sin 3x + c_2 \cos 3x$

Lagrange found a way to find the general soln. of the o.d.e. using the solution of $L[y] = 0$ but replacing c_1 and c_2 by $u(x)$ and $v(x)$.

Assume $y = u(x)\sin 3x + v(x)\cos 3x$

$$y' = u \cdot 3\cos 3x - v\, 3\sin 3x + u'\sin 3x + v'\cos 3x$$

before we find y'', consider that we will end up with $y'' + 9y = 9\sec^2 3x$ which will have 2 unknown functions, $u(x)$ and $v(x)$.

Since we will have a single eq. with 2 unknowns (functions), we expect infinitely many solutions. Lets pick one that makes our job easier:

$$u'\sin 3x + v'\cos 3x = 0$$

So

$$y' = 3u\cos 3x - 3v\sin 3x$$

$$y'' = -9u\sin 3x - 9v\cos 3x + 3u'\cos 3x - 3v'\sin 3x$$

$$y'' + 9y = 3u'\cos 3x - 3v'\sin 3x = 9\sec^2 3x$$

So
$$\begin{cases} u'\cos 3x - v'\sin 3x = 3\sec^2 3x \\ u'\sin 3x + v'\cos 3x = 0 \end{cases}$$

2 eq with 2 unknowns – solve

$$u' = \frac{\begin{vmatrix} 3\sec^2 3x & -\sin 3x \\ 0 & \cos 3x \end{vmatrix}}{\begin{vmatrix} \cos 3x & -\sin 3x \\ \sin 3x & \cos 3x \end{vmatrix}} = \frac{3\sec 3x}{\cos^2 3x + \sin^2 3x} = 3\sec 3x$$

$$v' = \frac{\begin{vmatrix} \cos 3x & 3\sec^2 3x \\ \sin 3x & 0 \end{vmatrix}}{1} = -3\sec^2 3x\sin 3x = -3\sec 3x\tan 3x$$

Now we integrate each to find $u(x)$, $v(x)$.

$$u(x) = \int 3\sec 3x\, dx = \ln|\sec 3x + \tan 3x| + c_1$$

$$v(x) = \int (-3)\sec 3x\tan 3x\, dx = -\sec 3x + c_2$$

So
$$y = \sin 3x\,(\ln|\sec 3x + \tan 3x| + c_1) + (-\sec 3x + c_2)\cos 3x$$

$$y = c_1\sin 3x + c_2\cos 3x + \sin 3x(\ln|\sec 3x + \tan 3x|) - 1$$

General Approach

Given $L[y] = y'' + p(x)y' + q(x)y = g(x)$

① Solve $L[y]=0$ to find $y_c = c_1 y_1 + c_2 y_2$

② Replace c_1 and c_2 by functions of x, $u_1(x)$, $u_2(x)$.

③ Require that $u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0 \quad \boxed{1}$
and substitute y into $L[y] = g(x)$ to get

$$u_1'(x)\, y_1'(x) + u_2'(x)\, y_2'(x) = g(x) \quad \boxed{2}$$

④ Solve $\boxed{1}$ & $\boxed{2}$ to find $u_1'(x)$, $u_2'(x)$

⑤ Integrate to find $u_1(x)$ & $u_2(x)$

General solution is $y = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

both sums where
one term is constant
(from integration)

Why not use this all the time
- integrals are sometimes difficult
- Can be tedious