3.6 Nonhomogeneous Equations : Method of Undetermined Coefficients

Now we return to eq. of the form

$$L[y] = y'' + p(x)y' + q(x)y = g(x)$$

If $Y_1(x)$ and $Y_2(x)$ are two solutions of $L[y] = g(x)$ then

$$L[Y_1] = L[Y_2] = g(x)$$

so $$L[Y_1] - L[Y_2] = g(x) - g(x) = 0$$

but since $L[\ ]$ is linear : $L[Y_1] - L[Y_2] = L[Y_1 - Y_2] = 0$
so $y = Y_1 - Y_2$ is a solution of the corresponding homogeneous eq. $L[y] = 0$.

$$Y_1 - Y_2 = \underbrace{c_1 y_1 + c_2 y_2}$$

fundamental set of solutions of $L[y]=0$

So

$$Y_1 = c_1 y_1 + c_2 y_2 + Y_2$$

Thm 3.6.2 The general solution of the nonhomogeneous eq $L[y] = g(x)$ can be written in the form

$$y = \phi(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x)$$

where y_1 and y_2 are a fundamental set of solutions of $L[y] = 0$, c_1 and c_2 are arbitrary constants, and Y is some specific solution of $L[y] = g(x)$.

i.e. "any"

So, Given $L[y] = g(x)$;

① Solve $L[y] = 0$ to find $y_c = c_1 y_1(x) + c_2 y_2(x)$

complementary solution

② Find a particular solution $y_p = Y$ of $L[y] = g(x)$

③ General solution then is $y(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x)$

We know how to do ①, at least when the eq. has constant coefficients so we will consider approaches to ②.

Ex. $y'' + 2y' - 3y = e^{-x}$

① $y'' + 2y' - 3 = 0 \rightarrow r^2 + 2r - 3 = 0 \rightarrow (r-1)(r+3) = 0$

$r = 1, -3$

$$y_c = c_1 e^{x} + c_2 e^{-3x}$$

② Need to find $y_p = Y$ which satisfies $y'' + 2y' - 3y = e^{-x}$

Make a guess → rhs has e^{-x} and we know that derivatives of this still have e^{-x}. So lets try $Y = Ae^{-x}$ and see if we can find A to make this work.

$$Y = Ae^{-x} \quad Y' = -Ae^{-x} \quad Y'' = Ae^{-x}$$

So

$$Ae^{-x} + 2\left[-Ae^{-x}\right] - 3\left[Ae^{-x}\right] = e^{-x}$$

$$(A-2A-3A)e^{-x}=e^{-x}$$

$$-4A=1 \rightarrow A=-1/4 \rightarrow Y=-\tfrac{1}{4}e^{-x}$$

(3) So, the general solution is

$$\boxed{y=c_1e^{x}+c_2e^{-3x}-\tfrac{1}{4}e^{-x}}$$

This approach is called the <u>method of undetermined coefficients</u>.

It assumes that we can make a good guess at a particular solution. For many $g(x)$ this may be possible: If $g(x)$ has exponential, sine or cosine, or polynomial functions of x, this method may work very well.

Ex Find Y to satisfy $y''+y'+y=\cos 2x$

assume $Y=A\cos 2x \rightarrow Y'=-2A\sin 2x \quad Y''=-4A\cos 2x$

$$-4A\cos 2x+2A\sin 2x+A\cos 2x=\cos 2x$$

↑
What can we do about this? nothing - does not work.

assume $Y=A\cos 2x+B\sin 2x$

$$Y'=-2A\sin 2x+2B\cos 2x$$
$$Y''=-4A\cos 2x-4B\sin 2x$$

$$-4A\cos 2x-4B\sin 2x-2A\sin 2x+2B\cos 2x+A\cos 2x+B\sin 2x=\cos 2x$$

$$(-4A+2B+A)\cos 2x+(-4B-2A+B)\sin 2x=\cos 2x$$

So $\quad -4A + 2B + A = 1 \qquad -3A + 2B = 1$

$\quad -4B - 2A + B = 0 \qquad -2A - 3B = 0 \rightarrow \quad B = -\frac{2}{3}A$

$-3A + 2(-\frac{2}{3}A) = 1$

$-9A - 4A = 3$

$$\boxed{\begin{aligned} A &= -3/13 \\ B &= 2/13 \end{aligned}}$$

$$Y_{(x)} = \frac{2}{13}\sin 2x - \frac{3}{13}\cos 2x.$$

How about "Find Y if $y'' + 9y = x^2 e^{-x}$"

See page 181 (8th ed)

we will assume $Y = (Ax^2 + Bx + C)e^{-x}$

$$Y' = (2Ax + B)e^{-x} - (Ax^2 + Bx + C)e^{-x}$$

$$Y'' = 2Ae^{-x} - (2Ax + B)e^{-x} - (2Ax + B)e^{-x} + (Ax^2 + Bx + C)e^{-x}$$

$$= 2Ae^{-x} - 2(2Ax + B)e^{-x} + (Ax^2 + Bx + C)e^{-x}$$

$$y'' + 9y = (2A - 4Ax - 2B + Ax^2 + Bx + C)e^{-x} + 9(Ax^2 + Bx + C)e^{-x} = x^2 e^{-x}$$

$$(Ax^2 + (-4A + B)x + 2A - 2B + C) + 9Ax^2 + 9Bx + 9C = x^2$$

$$\leftarrow 10Ax^2 + (10B - 4A)x + 10C + 2A - 2B = x^2$$

$$\begin{bmatrix} 10 & 0 & 0 \\ -4 & 10 & 0 \\ 2 & -2 & 10 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$10A = 1 \qquad A = 1/10$

$10B - 4A = 0 \qquad B = \frac{4}{10}A = \frac{4}{100}$

$10C + 2A - 2B = 0$

$10C = 2(\frac{4}{100}) - 2(\frac{10}{100}) = \frac{-12}{100}$

$C = -\frac{12}{1000}$

so $\quad Y_{(x)} = (\frac{1}{10}x^2 + \frac{4}{100}x - \frac{12}{1000})e^{-x} = \frac{100x^2 + 40x - 12}{1000}e^{-x}$

In general, if:

$g(x) =$	assume $Y =$
$\lambda e^{\mu x}$	$Ae^{\mu x}$
$\lambda_1 \sin\mu x + \lambda_2 \cos\mu x$	$A\sin\mu x + B\cos\mu x$
$\lambda_n x^n + \lambda_{n-1} x^{n-1} + \dots + \lambda_0$	$A_n x^n + A_{n-1} x^{n-1} + \dots + A_0$

If $g(x)$ is the product of any of these types, assume $Y(x)$ as the product of the corresponding functions.

Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	Assume $Y =$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_n$	① $t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)$
$P_n(t)e^{\alpha t}$	② $t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t \\ \cos\beta t\end{cases}$	③ $t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)e^{\alpha t}\cos\beta t + t^s(B_0 t^n + B_1 t^{n-1} + \dots + B_n)e^{\alpha t}\sin\beta t$

① s = number of times 0 is a root of $ay'' + by' + cy = 0$

② s = number of times α is a root of $ay'' + by' + cy = 0$

③ s = number of times $\alpha + i\beta$ is a root of $ay'' + by' + cy = 0$

Ex. Find the general solution of $y'' + 3y' + 2y = 2e^{-t}$

① Find complementary solution:

$$y'' + 3y' + 2y = 0$$
$$r^2 + 3r + 2 = 0$$
$$(r+1)(r+2) = 0$$

So $y_c = c_1 e^{-t} + c_2 e^{-2t}$

② Find particular solution $Y(t)$

First try would be $Y(t) = Ae^{-t}$
but this won't work since e^{-t} is a solution of the homogeneous problem. So we try $Y(t) = Ate^{-t}$

$$Y = Ate^{-t}$$
$$Y' = Ae^{-t} - Ate^{-t}$$
$$Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = Ate^{-t} - 2Ae^{-t}$$

$$Y'' + 3Y' + 2Y = Ate^{-t} - 2Ae^{-t} + 3(Ae^{-t} - Ate^{-t}) + 2Ate^{-t} = 2e^{-t}$$
$$= [A - 3A + 2A]te^{-t} + (-2A + 3A)e^{-t} = 2e^{-t}$$
$$= Ae^{-t} = 2e^{-t} \Rightarrow A = 2$$

③ General solution is $\boxed{y = c_1 e^{-t} + c_2 e^{-2t} + 2te^{-t}}$

Warning: If initial conditions are supplied they <u>must</u> <u>not</u> be used to find c_1 and c_2 until the general solution is found. [Need $Y(t)$ in general solution]