3.4 Complex Roots of the Characteristic Equation

Returning to $ay'' + by' + cy = 0$, recall that we found the characteristic eq. by assuming solutions of the form $y = e^{rx}$. This yields the characteristic equation

$$ar^2 + br + c = 0$$

which has the solution
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$ then there are 2 distinct real roots;
if $b^2 - 4ac = 0$ then there are real repeated roots and
if $b^2 - 4ac < 0$ the two roots are complex conjugates.

We restrict ourselves now to this last case, and write r_1 and r_2 in the form

$$r_1 = \lambda + i\mu \qquad \lambda = \frac{-b}{2a}$$
$$r_2 = \lambda - i\mu \qquad \mu = \frac{\sqrt{4ac - b^2}}{2a} = \sqrt{\frac{c}{a} - \lambda^2}$$

So our solutions are $y_1 = e^{(\lambda + i\mu)x}$ and $y_2 = e^{(\lambda - i\mu)x}$.
or

$$y_1 = e^{\lambda x} e^{i\mu x} \qquad y_2 = e^{\lambda x} e^{-i\mu x}$$

What does $e^{i\mu x}$ mean? Lets use the Maclaurin Series (Taylor Series about 0)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

so
$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$
$$= \cos x + i \sin x$$

using Maclaurin series for $\cos x$ and $\sin x$

Thus, it makes sense to define
$$e^{ix} = \cos x + i \sin x$$

We also need
$$e^{-ix} = e^{i(-x)} = \cos(-x) + i\sin(-x) = \cos x - i \sin x$$

Euler - Identities

So we have
$$e^{ix} = \cos x + i \sin x \qquad e^{-ix} = \cos x - i \sin x$$

Aside

The equation $e^{i\pi} + 1 = 0$ is true and uses the 5 fundamental constants $e, i, \pi, 1, 0$.

Back to our problem...

So $y_1 = e^{\lambda x}[\cos \mu x + i \sin \mu x]$ $\qquad y_2 = e^{\lambda x}[\cos \mu x - i \sin \mu x]$

but $y_1 + y_2 = 2e^{\lambda x}\cos \mu x$ $\qquad y_1 - y_2 = 2i\, e^{\lambda x} \sin \mu x$

also form a fundamental set of solutions. Thus we know that the linear combination

$$y = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$$

is a general solution of our o.d.e.

Ex $y''-2y'+2y=0 \rightarrow r^2-2r+2=0 \rightarrow r=\frac{2\pm\sqrt{4-8}}{2} = 1\pm i$

general solution is $y = c_1 e^x \cos x + c_2 e^x \sin x$

or

$$y = e^x(c_1\cos x + c_2 \sin x)$$

Ex $y''+4y=0 \quad y(0)=0 \quad y'(0)=1$

$r^2+4=0 \rightarrow r=\pm 2i \rightarrow y = e^0(c_1\cos 2x + c_2 \sin 2x)$
$= c_1\cos 2x + c_2 \sin 2x$

Now match I.C.

$0 = c_1\cos(0) + c_2\sin(0) = c_1 \rightarrow c_1 = 0$

so

$y = c_2 \sin 2x \rightarrow y' = 2c_2\cos 2x$

$y'(0) = 1 = 2c_2\cos(0) = 2c_2 \rightarrow c_2 = 1/2$

so

$$\boxed{y(x) = \tfrac{1}{2}\sin 2x}$$

Summary

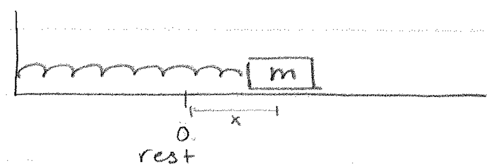
If the characteristic eq has the complex roots $r = \lambda \pm i\mu$
assume a general solution of the form

$$y = e^{\lambda x}[c_1\cos\mu x + c_2 \sin \mu x]$$

and determine c_1 and c_2 using the initial conditions

NOTE: If $\lambda<0$ solutions decay to zero, if $\lambda>0$ solutions oscillate with increasing amplitude. If $\lambda=0$ solutions oscillate periodically. IMAG PART $\rightarrow$ OSC. REAL PART $\rightarrow$ MAGNITUDE OF LONG TERM BEHAVIOR

Ex Simple Harmonic Oscillator



Force due to spring is $-kx$, negative since it acts in the opposite direction of the displacement

$$F = ma = -kx \rightarrow m\ddot{x} = -kx \rightarrow m\ddot{x} + kx = 0$$

double dots denote derivative wrt time.

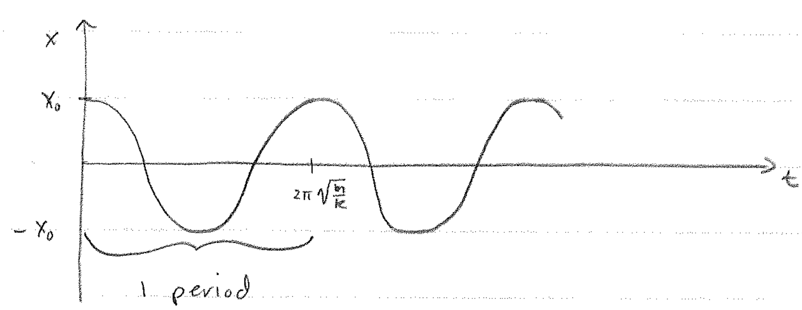
We get $mr^2 + k = 0 \rightarrow r^2 = -k/m \rightarrow r = \pm i\sqrt{\frac{k}{m}}$

so $\mu = \sqrt{k/m}$, $\lambda = 0$.

$$x = c_1 \cos\sqrt{\frac{k}{m}}\,t + c_2 \sin\sqrt{\frac{k}{m}}\,t$$

If $x(0) = x_0$ and $\dot{x}(0) = 0$ we find

$$x = x_0 \cos\sqrt{\frac{k}{m}}\,t$$



occurs when $\sqrt{\frac{k}{m}}\,t = 2\pi \rightarrow t = \frac{2\pi}{\sqrt{\frac{k}{m}}}$

So the period of oscillation is $T = 2\pi\sqrt{\frac{m}{k}}$ and the frequency, $1/T$ is $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$