3.2 Fundamental Solutions of Linear Homogeneous Equations

We now seek to answer some basic questions regarding 2^{nd} order linear o.des.

1) When does the eq. have a solution?
2) If we find a solution – are there others?

Begin by introducing the Second Order Linear Differential Operator

$$L[\phi] = \phi'' + p(x)\phi' + q(x)\phi$$

Where p and q are continuous on open interval $I: \alpha < x < \beta$ and ϕ is twice-differentiable on I.

So, the general Second order linear Initial Value problem is

$$\left.\begin{array}{ll} \text{Find } y \text{ s.t.} & L[y] = y'' + p(x)y' + q(x)y = g(x) \\ \text{Subject to} & y(x_0) = y_0 \\ & y'(x_0) = y_0' \end{array}\right\} \quad (*)$$

Thm 3.2.1

If p, q and g from $(*)$ are continuous on an open Interval I, then there exists exactly one solution $y = \phi(x)$ of $(*)$, and the solution exists throughout I.

Note that the o.d.e must be in the form of $(*)$, with the coefficient of y'' equal to 1.

Thm 3.2.2 (principle of superposition) (linear combination)

If y_1 and y_2 are two solutions of the differential equation

homogeneous!

$$L[y] = y'' + p(x)y' + q(x)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 & c_2.

Proof

$$L[c_1y_1 + c_2y_2] = \frac{d^2}{dx^2}(c_1y_1 + c_2y_1) + p(x)\frac{d}{dx}(c_1y_1 + c_2y_2) + q(x)(c_1y_1 + c_2y_2) = 0$$

$$= c_1y_1'' + c_2y_2'' + p(x)c_1y_1' + p(x)c_2y_2' + q(x)c_1y_1 + q(x)c_2y_2 = 0$$

$$= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2) = 0$$

$$= c_1L[y_1] + c_2L[y_2] = 0$$

$$= c_1(0) + c_2(0) = 0 \checkmark$$

This works because L is a linear operator.

So ...

p, q, g continuous on I

Assuming certain things about the d.e., we know

1) that a unique solution will exist, given specific initial conditions

2) that a linear combination of functions which satisfy the d.e. will also satisfy the d.e.

homogeneous

Given suitable I.C. and properties of the diff. eq, we now consider the question

"Can I find c_1 and c_2 such that $y = c_1 y_1 + c_2 y_2$ satisfies the initial conditions?"

1. $y(x_0) = y_0$ so $y_0 = c_1 y_1(x_0) + c_2 y_2(x_0)$
2. $y'(x_0) = y_0'$ so $y_0' = c_1 y_1'(x_0) + c_2 y_2'(x_0)$

2 eq. with 2 unknown

$$y_1(x_0)\, c_1 + y_2(x_0)\, c_2 = y_0$$
$$y_1'(x_0)\, c_1 + y_2'(x_0)\, c_2 = y_0'$$

using Cramers rule we find

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(x_0) \\ y_0' & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(x_0) & y_0 \\ y_1'(x_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}$$

which say that c_1 and c_2 do indeed exist provided the determinant in the denominator is not zero.

$$W = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)\, y_2'(x_0) - y_1'(x_0)\, y_2(x_0)$$

This determinant is called the "Wronskian"

Thm 3.2.3 follows... (page 134) (6th?) (pg 141 7th)

Main Thrust → if $W \neq 0$ at x_0 where the initial conditions are supplied, then c_1 and c_2 can be fund for which $y = c_1 y_1 + c_2 y_2$ satisfies (*) (the d.e. and the i.c.s).

Ex Recall

$$y'' - 4y = 0 \quad \rightarrow \quad y = c_1 e^{2x} + c_2 e^{-2x}$$

$$\left.\begin{array}{l} y(0) = 1 \\ y'(0) = 0 \end{array}\right\} \rightarrow \quad c_1 = \tfrac{1}{2}, \quad c_2 = 1/2$$

here $y_1 = e^{2x}$; $y_2 = e^{-2x}$

$$W = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} e^0 & e^0 \\ 2e^0 & -2e^0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -2 - 2 = -4$$

not zero

$$c_1 = \frac{\begin{vmatrix} 1 & y_2(0) \\ 0 & y_2'(0) \end{vmatrix}}{W} = \frac{1(-2) - 0}{-4} = \frac{1}{2} \checkmark$$

$$c_2 = \frac{\begin{vmatrix} y_1(0) & 1 \\ y_1'(0) & 0 \end{vmatrix}}{W} = \frac{0 - 2}{-4} = \frac{1}{2} \checkmark$$

→ The solution, $y = c_1 y_1 + c_2 y_2$, is called the <u>general solution</u> since <u>any</u> solution of the o.d.e. is contained within this family of functions

i.e. a linearly independent set of solutions that spans the set of all solutions

→ If $W \neq 0$, y_1 and y_2 form a <u>fundamental set of solutions</u>.

THM 3.2.4

If y_1 and y_2 are two solutions of the differential equation $L[y] = y'' + p(x)y' + q(x)y = 0$, and if there is a point x_0 where the Wronskian of y_1 and y_2 is nonzero, then the family of solutions $y = c_1y_1 + c_2y_2$ with arbitrary c_1 and c_2, includes every solution of the o.d.e.

Proof:

not a particularly interesting proof

Assume $\phi(x)$ is <u>any</u> function which satisfies o.d.e. and initial conditions (*). Thus $\phi(x_0) = y_0$ and $\phi'(x_0) = y_0'$.

Now show that $\phi(x)$ is contained in $y = c_1y_1 + c_2y_2$.

Since by Thm 3.2.2 $y = c_1y_1 + c_2y_2$ is a solution of the o.de. and since $W = y_1(x_0)\,y_2'(x_0) - y_1'(x_0)\,y_2(x_0) \neq 0$ we know by Thm 3.2.3 that there exist c_1 and c_2 which allow y to solve (*).

By Thm 3.2.1, if $\phi(x)$ and $y = c_1y_1 + c_2y_2$ both solve (*) then $\phi(x) = c_1y_1 + c_2y_2$ since the solution is unique.

Thus ϕ is contained in $y = c_1y_1 + c_2y_2$. ■

Please note that while Thm 3.2.4 says y_1 and y_2, as fundamental solutions, form a <u>basis</u> for all solutions, it is possible that solution <u>appear</u> not to depend on these functions

<u>Ex</u> Recall $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 0$ had the solution

$$y = \tfrac{1}{2}e^{2x} + \tfrac{1}{2}e^{-2x}.$$

Well, $y = c_1 \cosh 2x + c_2 \sinh 2x$ also satisfies the o.d.e.

$$\cosh u = \frac{e^u + e^{-u}}{2} \qquad \sinh u = \frac{e^u - e^{-u}}{2}$$

Finding c_1 & c_2, we match i.c.s and get

$y = \cosh 2x$ is the solution.