3.1 Homogeneous Equations with Constant Coefficients

Second Order Equations - now have terms with y'' as well as y' and y.

Genenera/ Form : $y'' = f(x, y, y')$.

If the eq. can be written in the form

$$y'' + p(x)y' + q(x)y = g(x)$$

it is a <u>linear 2nd order o.d.e.</u> This is sometimes expressed as

$$P(x)y'' + Q(x)y' + R(x)y = G(x)$$

where the coefficient of y'' is not one. This can always be reduced to the first form whenever $P(x) \neq 0$.

Nearly all our study of 2nd order eq. will be restricted to linear eq. Many interesting problems, however, can at least be approximated by linear equations.

We begin by considering equations for which $G(x) = 0$:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

This is a <u>homogeneous equation</u>. If $G(x)$ is nonzero the equation is called <u>nonhomogeneous.</u>

Now consider homogeneous equations with constant coefficients.

$$\begin{cases} ay'' + by' + cy = 0. \\ y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases}$$

here we see 2 initial conditions - we will need both.

Ex
$$\begin{cases} y'' - 4y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

What function do you know which has itself as its second derivative? e^x

Is $y = e^x$ a solution? $y'' - 4y = e^x - 4e^x = e^x(1-4)$
$= 3e^x \neq 0$

No

How about $y = e^{2x}$, $y'' = 4e^{2x}$ so
$y'' - 4y = 4e^{2x} - 4e^{2x} = 0$
YES!

So $y_1 = e^{2x}$ is a solution. Are there any others?

$y_2 = e^{-2x}$ is also a solution.

If $y_1 = e^{2x}$ is a solution, then $c_1 y_1$ is also a solution

$$(c_1 y_1)'' - 4(c_1 y_1) = c_1 y_1'' - c_1 4 y_1 = c_1 (y_1'' - 4y_1) = c_1 (0) = 0$$

ditto for y_2.

Actually, any linear combination of y_1 and y_2 is a solution

$$y = c_1 y_1 + c_2 y_2$$
$$y'' = c_1 y_1'' + c_2 y_2'' \qquad \text{since } \frac{d^2}{dx^2} \text{ is a linear operator}$$

so

$$\begin{aligned} y'' - 4y &= c_1 y_1'' + c_2 y_2'' - 4(c_1 y_1 + c_2 y_2) \\ &= c_1[y_1'' - 4y_1] + c_2[y_2'' - 4y_2] \\ &= c_1(0) + c_2(0) = 0 \end{aligned}$$

We call $y = c_1 e^{2x} + c_2 e^{-2x}$ the <u>general solution</u> since every solution of the o.d.e. is included in it.

Now, about those constants...

Using the initial conditions

$$y(0) = 1 \quad \rightarrow \qquad 1 = c_1 e^0 + c_2 e^0 = c_1 + c_2$$

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x}$$
$$y'(0) = 0 \quad \rightarrow \quad 0 = 2c_1 e^0 - 2c_2 e^0$$
$$0 = c_1 - c_2 \quad \rightarrow \quad c_1 = c_2$$

Since $c_1 = c_2$ and $c_1 + c_2 = 1$ we find $c_1 = c_2 = \frac{1}{2}$

so the unique solution to the o.d.e. is

$$y = \tfrac{1}{2}e^{2x} + \tfrac{1}{2}e^{-2x} \quad \text{or} \quad y = \tfrac{1}{2}(e^{2x} + e^{-2x}).$$

$$\text{or} \quad y = \cosh(2x)$$

In general the equation

$$ay'' + by' + cy = 0$$

with the initial conditions

$$y(x_0) = y_0 \ , \quad y'(x_0) = y_0'$$

can be solved in a similar way.

Let us look for solutions of the form $y = e^{rx}$:

$$y = e^{rx}$$
$$y' = re^{rx} \quad \rightarrow \quad ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$
$$y'' = r^2e^{rx} \qquad (ar^2 + br + c)e^{rx} = 0$$
$$ar^2 + br + c = 0 \quad \text{since } e^{rx} \neq 0$$

This is the characteristic equation.

We can solve this for r, and one of three things will occur

a) 2 distinct real roots

b) real repeated roots

c) complex conjugates

For now we only consider (a).

After finding roots r_1 & r_2, $r_1 \neq r_2$, we can construct the general solution:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Finally we can use the initial conditions to solve for the constants c_1 & c_2.

Ex
$$\begin{cases} y''+4y'+3y=0 \\ y(0)=2 \\ y'(0)=-1 \end{cases}$$

assume $y=e^{rx}$ $\rightarrow$ $r^2e^{rx}+4re^{rx}+3e^{rx}=0$

$r^2+4r+3=0$

$(r+1)(r+3)=0 \rightarrow r=-1, r=-3$

So general solution is

$$y=c_1e^{-x}+c_2e^{-3x}$$

Find c_1, c_2: $y(0)=2=c_1e^0+c_2e^0= c_1+c_2$

$y'(x) = -c_1e^{-x}-3c_2e^{-3x}$

$y'(0)=-1=-c_1-3c_2 \rightarrow c_1=1-3c_2$

$2=(1-3c_2)+c_2 = 1-2c_2$

$-\frac{1}{2}=c_2$ $\quad c_1=1-3(-\frac{1}{2})=5/2$

$$\boxed{y=\frac{5}{2}e^{-x}-\frac{1}{2}e^{-3x}}$$

check: $y'=-\frac{5}{2}e^{-x}+\frac{3}{2}e^{-3x}$

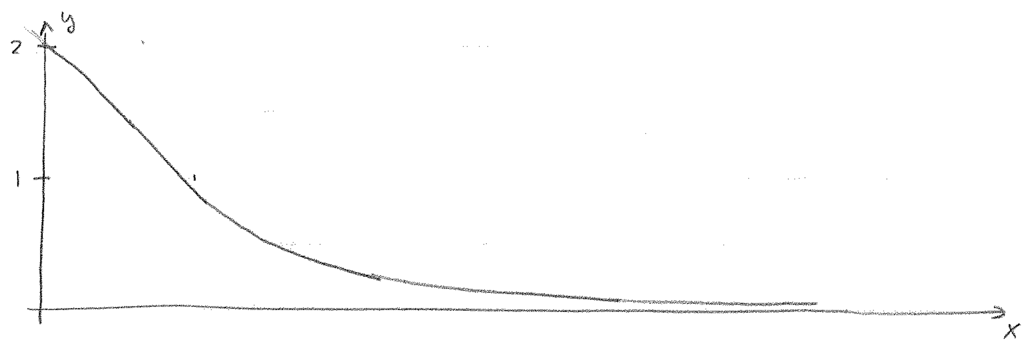
$y''=\frac{5}{2}e^{-x}-\frac{9}{2}e^{-3x}$

$$\left(\frac{5}{2}e^{-x}-\frac{9}{2}e^{-3x}\right)+4\left(-\frac{5}{2}e^{-x}+\frac{3}{2}e^{-3x}\right)+3\left(\frac{5}{2}e^{-x}-\frac{1}{2}e^{-3x}\right)=0 \quad \checkmark$$

We can sketch this as well

- $y(0) = 2$
- $y'(0) = -1$
- $\lim_{x \to \infty} y = 0$
- since $e^{-3x} \le e^{-x} \le 1$ when $x \geq 0$

  we know $\frac{5}{2} e^{-x} > -\frac{3}{2} e^{-3x}$ so $y' < 0$ if $x \geq 0$



as $x \to \infty$ we see $y \to 0$ from above.