2.6 Exact Equations and Integrating Factors

We saw before that a 1^{st} order o.d.e can be expressed in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (*)$$

If the equation was linear it become

$$M_1(x)\,y + N(x)\frac{dy}{dx} = M_2(x) \qquad \text{where } M(x,y) = M_1(x)\,y - M_2(x)$$

and we used an integrating factor to obtain

$$[\mu(x)\,y]' = \mu(x)\,M_2(x)/M_1(x), \quad \mu(x) = e^{\int \frac{N(x)}{M_1(x)}dx}$$

Using the product rule backwards.

Lets try a similar approach with $(*)$. Recall the def. of the <u>total derivative</u>.

$$\frac{d}{dx}\psi(x,y) = \frac{\partial\psi}{\partial x}\frac{dx}{dx} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

If we can find $\psi(x,y)$ so that

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

then we can write $\frac{d\psi}{dx} = 0$ so $\psi = c$ and we will have solved the o.d.e. implicitly for y.

$\partial M / \partial y = M_y$

Our task is to find $\Psi(x,y)$ so that

$$\frac{\partial \Psi}{\partial x} = M(x,y) \quad , \quad \frac{\partial \Psi}{\partial y} = N(x,y)$$

$$\frac{\partial \Psi}{\partial x} = M(x,y) \implies \Psi(x,y) = \int M(x,y)\,dx + h(y)$$

like a constant of int.

now diff wrt y

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left[\int M\,dx + h(y)\right] = \int \frac{\partial M}{\partial y}\,dx + h'(y)$$

so

$$N(x,y) = \int \frac{\partial M}{\partial y}\,dx + h'(y)$$

or

$$h'(y) = N(x,y) - \int \frac{\partial M}{\partial y}\,dx$$

This says that the RHS must only be a function of y (since the LHS is only a function of y) so $\frac{\partial}{\partial x}(\text{RHS})$ must be zero

$$\frac{\partial}{\partial x}\left[N - \int \frac{\partial M}{\partial y}\,dx\right] = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

or

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \qquad (**)$$

If $(**)$ is true, then we need only find $h(y)$ and then

$$\Psi(x,y) = \int M\,dx + h(y) = c$$

yields the solution.

<u>Ex</u> $2xy^2 + (2x^2y - 3y^2)\frac{dy}{dx} = 0$

$M(x,y) = 2xy^2 \qquad N(x,y) = 2x^2y - 3y^2$

Does $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$? $\left.\begin{array}{l}\frac{\partial M}{\partial y} = 4xy \\ \frac{\partial N}{\partial x} = 4xy\end{array}\right\}$ same !

So $\psi = \int M(x,y)\,dx + h(y) \quad \left[\text{or } \psi = \int N(x,y)\,dy + g(x)\right]$

$= \int 2xy^2 dx + h(y)$

$= x^2y^2 + h(y)$

but

$N(x,y) = \frac{\partial \psi}{\partial y}$ so $2x^2y - 3y^2 = \frac{\partial}{\partial y}\left[x^2y^2 + h(y)\right]$

$= 2x^2y + h'(y)$

$\therefore -3y^2 = h'(y)$

$-y^3 = h + c$ c can be dropped since we'll account for it in the final solution

$\therefore \quad \psi = x^2y^2 - y^3.$

Since $\frac{d}{dx}\psi = 0$ we have $\psi = x^2y^2 - y^3 = c$

$$\boxed{x^2y^2 - y^3 = c}$$

Equations which have the property $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ are called <u>Exact</u>.

Basic techique

1. Put o.d.e into $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ form
2. Check for exactness, i.e. does $M_y = N_x$?
3. Integrate M wrt x and add an arbitrary function of y to get $\psi = \int M\,dx + h(y)$
4. Use $\frac{\partial \psi}{\partial y} = N$ to find $h(y)$.
5. Solution is $\psi = c$.

Ex #16. $(ye^{2xy} + x)\,dx + bxe^{2xy}\,dy = 0$

$$M = ye^{2xy} + x \qquad N = bxe^{2xy}$$

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xye^{2xy}$$

$$\frac{\partial N}{\partial x} = be^{2xy} + b2xye^{2xy}$$

need $b = 1$

so D.E. is

$$(ye^{2xy} + x) + (xe^{2xy})\frac{dy}{dx} = 0$$

$$\psi = \int (ye^{2xy} + x)\,dx + h(y) =$$

$$= \frac{y}{2y}e^{2xy} + \frac{x^2}{2} + h(y) = \frac{1}{2}e^{2xy} + \frac{x^2}{2} + h(y)$$

$$N = \frac{\partial \psi}{\partial y} = xe^{2xy} + h'(y) = xe^{2xy} \quad \rightarrow h = \text{constant}$$

$$\therefore \quad \frac{1}{2}e^{2xy} + \frac{x^2}{2} = c \quad \text{or} \quad \boxed{e^{2xy} + x^2 = c}$$

$$y = \frac{1}{2x}\ln|c - x^2|$$

If the eq. is not exact, we can try and find an integrating factor $\mu(x,y)$ which makes it exact.

$$\mu(x,y)\, M(x,y) + \mu(x,y)\, N(x,y)\, \frac{dy}{dx} = 0$$

We need $$\frac{\partial}{\partial y}\left[\mu M\right] = \frac{\partial}{\partial x}\left[\mu N\right]$$

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

or

$$N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y} + \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]\mu = 0$$

In general, this p.d.e. is at least as hard to solve as the original o.d.e. If, however, $\mu(x,y)$ is a function of x only then

$$N \frac{d\mu}{dx} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu = 0$$

or

$$\frac{d\mu}{dx} + \frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu = 0 \qquad \boxed{\mu(x)}$$

which <u>may</u> be solvable for μ.

for $\mu = \mu(y)$

$$\frac{d\mu}{dy} - \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu = 0$$

Ex. $2xy + (2x^2 - 3y)\frac{dy}{dx} = 0$ $M = 2xy$

$N = 2x^2 - 3y$

is it exact?

$\frac{\partial M}{\partial y} = 2x$ $\frac{\partial N}{\partial x} = 4x$

no.

is $\mu(x)$ an integrating factor? Is this easy to solve?

$$\frac{d\mu}{dx} + \frac{1}{2x^2-3y}(4x-2x)\mu = 0$$

$$\frac{d\mu}{dx} + \frac{2x}{2x^2-3y}\mu = 0$$

since $\mu(x)$ cannot depend on y this won't work!

how about $\mu(y)$? $\frac{d\mu}{dy} - \frac{1}{2xy}(4x-2x)\mu = 0$

$$\frac{d\mu}{dy} - \frac{1}{y}\mu = 0 \rightarrow \frac{d\mu}{\mu} = \frac{dy}{y}$$

$\ln\mu = \ln y + c$

$\mu = e^c y = cy$

take $c=1$

So $\mu = y$ is an int. factor

thus

$2xy^2 + (2x^2y - 3y^2)\frac{dy}{dx} = 0$ is exact.

(our example before).