2.3 Applications of First Order linear Equations

Mathematical Modelling

Three steps in process from statement of physical problem to solve to its solution.

1. <u>Translation</u>
   - understand the physical problem
   - give names (symbols) to unknown quantities
   - develop one or more equations which describe the physical phenomena. must also usually determine values of unknown quantities at one or more particular points.

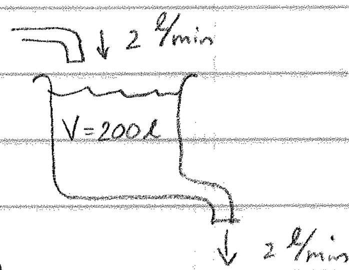
2. <u>Solve</u>
   - analytical or numerical ?
   - approximate or exact solution?
   - bring any and all information about the physical problem to bear

3. <u>Interpretation</u>
   - does it make sense?
   - is it unique?
   - is it stable?

Often, step 3 may lead back to step 1.

2.3 Modeling with First order Equations

Ex. # 1



Dye concentration initially 1 g/l

Assume Perfect Mixing

Let $c(t)$ be the concentration at time t, so

$q(t) = Vc(t)$ is the quantity of dye in tank at time t.

dq/dt has units g/min and is the rate at which the quantity of dye in the tank is changing.

$$\frac{dq}{dt} = \begin{bmatrix}\text{quantity in}\\ \text{per unit time}\end{bmatrix} - \begin{bmatrix}\text{quantity out}\\ \text{per unit time}\end{bmatrix}$$

$$0 \text{ g/min} \qquad (2 \text{ l/min})(c(t) \text{ g/l})$$

$$2c \text{ g/min}$$

Express $c(t)$ in terms of $q(t)$ to obtain

$$\frac{dq}{dt} = -2\frac{q(t)}{V} = -2\frac{q(t)}{200} \Rightarrow \frac{dq}{dt} = \frac{q(t)}{100}$$

Solve using separation of variables

$$\frac{dq}{q} = -\frac{1}{100}dt \rightarrow \ln|q| = -\frac{t}{100} + k$$

So

$$q(t) = Ke^{-t/100}$$

Initial Condition : $c(0) = 1$ g/l so $q(0) = 200$ g

$$200 = Ke^{0} \rightarrow K = 200$$

So

$$\boxed{q(t) = 200e^{-t/100}}$$

Find when $q(t) = 0.01\,q(0)$

$$(0.01)(200) = 200e^{-t/100}$$

$$0.01 = e^{-t/100}$$

$$\ln 0.01 = -t/100$$

$$t = -100\ln 0.01$$

$t = 460.52$ min or

7 hours, 40.52 minutes

#10 (7th edition) Homebuyer wants payments of \$800/month.

APR is 9% Term is 20 years

Assume continuous compounding @ continuous payments

Find maximum amount he can borrow

D.E. will be

$$\frac{dS}{dt} = rS + k$$

rate of decay of principle

will be proportional to current balance plus a payment (negative)

$$\frac{dS}{dt} - rS = k \qquad \mu = e^{-\int r\,dt} = e^{-rt}$$

$$(e^{-rt}S)' = e^{-rt}k$$

$$e^{-rt}S = k\int e^{-rt}\,dt$$

$$= -\frac{k}{r}e^{-rt} + C$$

$$\therefore \quad S = -\frac{k}{r} + Ce^{rt}$$

Let S_0 be amount borrowed. Then $S = S_0$ when $t=0$

$$S_0 = -\frac{k}{r} + Ce^{0} \quad \rightarrow \quad C = S_0 + k/r$$

$$\therefore \quad S = S_0e^{rt} + \frac{k}{r}(e^{rt}-1)$$

After 20 years $S=0$. $k = -(12)(800) = -9600$ annual payments

$$0 = S_0e^{(0.09)(20)} + \frac{-9600}{0.09}(e^{(0.09)(20)} - 1)$$

$S_0 = 89{,}034.79$ Max. amount to borrow is \$89,034.79

Total interest is $20 \cdot 12 \cdot 800 - 89034.79$

$= \$102{,}965.21$

#19. Lake Pollution

Lake volume $\Rightarrow V$ constant

Amount of Pollutant $\to Q(t)$ varies

Want to get DE with either amount $Q(t)$ or concentration $c(t) = Q(t)/V$. I choose $Q(t)$

So {Rate of change of amount at time t} =
{rate of pollutants entering} −
{rate of pollutant leaving}

Basic assumption − <u>Perfect mixing</u>

Rate of change of amount is $\frac{d}{dt} Q(t)$

rate of pollutants entering is $P + kr$
rate of pollutants leaving is $\frac{Q(t)}{V} r$

So

$$\frac{d}{dt} Q = P + kr - \frac{Q}{V} r$$

or

$$\begin{cases} \frac{d}{dt} Q + \frac{r}{V} Q = P + kr \\ \text{Some initial condition to be imposed later} \end{cases}$$

1^{st} order linear DE $\to$ integrating factors

Use $Q(t) = \frac{1}{\mu(t)} \int \mu(t)\,(P+kr)\,dt$; $\mu(t) = e^{\int \frac{r}{V} dt} = e^{\frac{rt}{V}}$

$$Q(t) = e^{-\frac{rt}{V}} (P+kr) \int e^{\frac{rt}{V}} dt$$

$$= e^{-\frac{rt}{V}} (P+kr) \left[\frac{V}{r} e^{\frac{rt}{V}} + c \right]$$

$$Q(t) = \frac{V}{r}(P+kr) + c\, e^{-\frac{rt}{V}}$$

↑ includes $(p+kr)$ factor

Part (a) wants $c(t)$ so we convert to concentration by dividing by V

$$c(t) = \frac{P}{r} + k + \bar{C} e^{-\frac{rt}{V}}$$

and use $c(0) = C_0$ to find constant

$$C_0 = \frac{P}{r} + k + \bar{C} \rightarrow \bar{C} = C_0 - \left(\frac{P}{r} + k\right)$$

So

$$\boxed{c(t) = \frac{P}{r} + k + \left[C_0 - \left(\frac{P}{r} + k\right)\right] e^{-\frac{rt}{V}}}$$

As $t \to \infty$ the concentration $\rightarrow \frac{P}{r} + k$

$$\boxed{\lim_{t \to \infty} c(t) = \frac{P}{r} + k}$$

Part (b)

if $k = P = 0$ then $c(t) = C_0 e^{-\frac{rt}{V}}$

To find time until $c(t) = \frac{1}{2} C_0$

$$\frac{1}{2} C_0 = C_0 e^{-\frac{rT}{V}} \rightarrow \ln \frac{1}{2} = \frac{-rT}{V} \rightarrow \boxed{T = \frac{V}{r} \ln 2}$$

2.3

time until $C(t) = \frac{1}{10} C_0$

$$\frac{1}{10} C_0 = C_0 e^{-\frac{rT}{V}}$$

$$-\ln 10 = -\frac{rT}{V}$$

$$\boxed{T = \frac{V}{r} \ln 10}$$

Part (c)

Lake superior $V = 12.2 \times 10^{+3}$ km^3 $r = 65.2$ km^3/year

$$T_{\frac{1}{10}} = \frac{12.2 \times 10^3}{65.2} \ln 10 \simeq \underline{431 \text{ years}}$$

Lake Michigan $V = 4.9 \times 10^3$ km $r = 158$ km^3/yr

$$T_{\frac{1}{10}} = \frac{4.9 \times 10^3}{158} \ln 10 \simeq \underline{71.4 \text{ years}}$$

Lake Erie $V = 0.46 \times 10^3$ km^3 $r = 175$ km^3/yr

$$T_{\frac{1}{10}} = \frac{0.46 \times 10^3}{175} \ln 10 \simeq \underline{6.05 \text{ years}}$$

Lake Ontario $V = 1.6 \times 10^3$ km^3 $r = 209$ km^3/yr

$$T_{\frac{1}{10}} = \frac{1.6 \times 10^3}{209} \ln 10 \simeq \underline{17.6 \text{ years}}$$

Ex Snowplow Problem (Nagle - Saff, pg 66)

Begins snowing hard in morning and snows steadily throughout the day. Snowplow starts at 9Am and by 11Am it has cleared 2 miles. By 1pm it has cleared an additional mile. What time did it start snowing?

Assumptions

1. Snowing at a constant rate
2. plow speed is inversely proportional to snow depth

Statement

$s(t)$ = distance plowed at time t. (miles)

r = constant rate of snowfall

key assumption

$$\{\text{plow speed}\} = \frac{k}{\{\text{snow depth}\}}$$ k = constant of proportionality

$\{\text{snow depth}\} = r(t - t_0)$ t_0 = time snow began

$\{\text{plow speed}\} = \dfrac{ds}{dt}$

So $$\boxed{\frac{ds}{dt} = \frac{k}{r(t-t_0)}}$$ our diff. eq.

Solution

$$ds = \frac{k\,dt}{r(t-t_0)}$$

$$S(t) = \frac{k}{r}\ln(t-t_0) + C$$

3 unknowns $\frac{k}{r}$, t_0, C

3 conditions (data points)

At $t = 9$ $S = 0$

$t = 11$ $S = 2$

$t = 13$ $S = 3$

So $S(9) = 0 = \frac{k}{r}\ln(9-t_0) + C \rightarrow C = -\frac{k}{r}\ln(9-t_0)$

$$S(t) = \frac{k}{r}\ln(t-t_0) - \frac{k}{r}\ln(9-t_0) = \frac{k}{r}\ln\frac{t-t_0}{9-t_0}$$

$$S(11) = 2 = \frac{k}{r}\ln\frac{11-t_0}{9-t_0} \rightarrow \frac{k}{r} = \frac{2}{\ln\frac{11-t_0}{9-t_0}}$$

$$S(t) = \frac{2}{\ln\frac{11-t_0}{9-t_0}}\ln\left(\frac{t-t_0}{9-t_0}\right)$$

$$S(13) = 3 = \frac{2}{\ln\frac{11-t_0}{9-t_0}}\ln\left(\frac{13-t_0}{9-t_0}\right)$$

$$3\ln\frac{11-t_0}{9-t_0} = 2\ln\frac{13-t_0}{9-t_0}$$

$$\left(\frac{11-t_0}{9-t_0}\right)^3 = \left(\frac{13-t_0}{9-t_0}\right)^2$$

$$(11-t_0)^3 = (13-t_0)^2(9-t_0)$$

$$1331 - 363t_0 + 33t_0^2 - t_0^3 = 1521 - 403t_0 + 35t_0^2 - t_0^3$$

$$0 = 190 - 40t_0 + 2t_0^2$$

$$t_0^2 - 20t_0 + 95 = 0$$

2.3

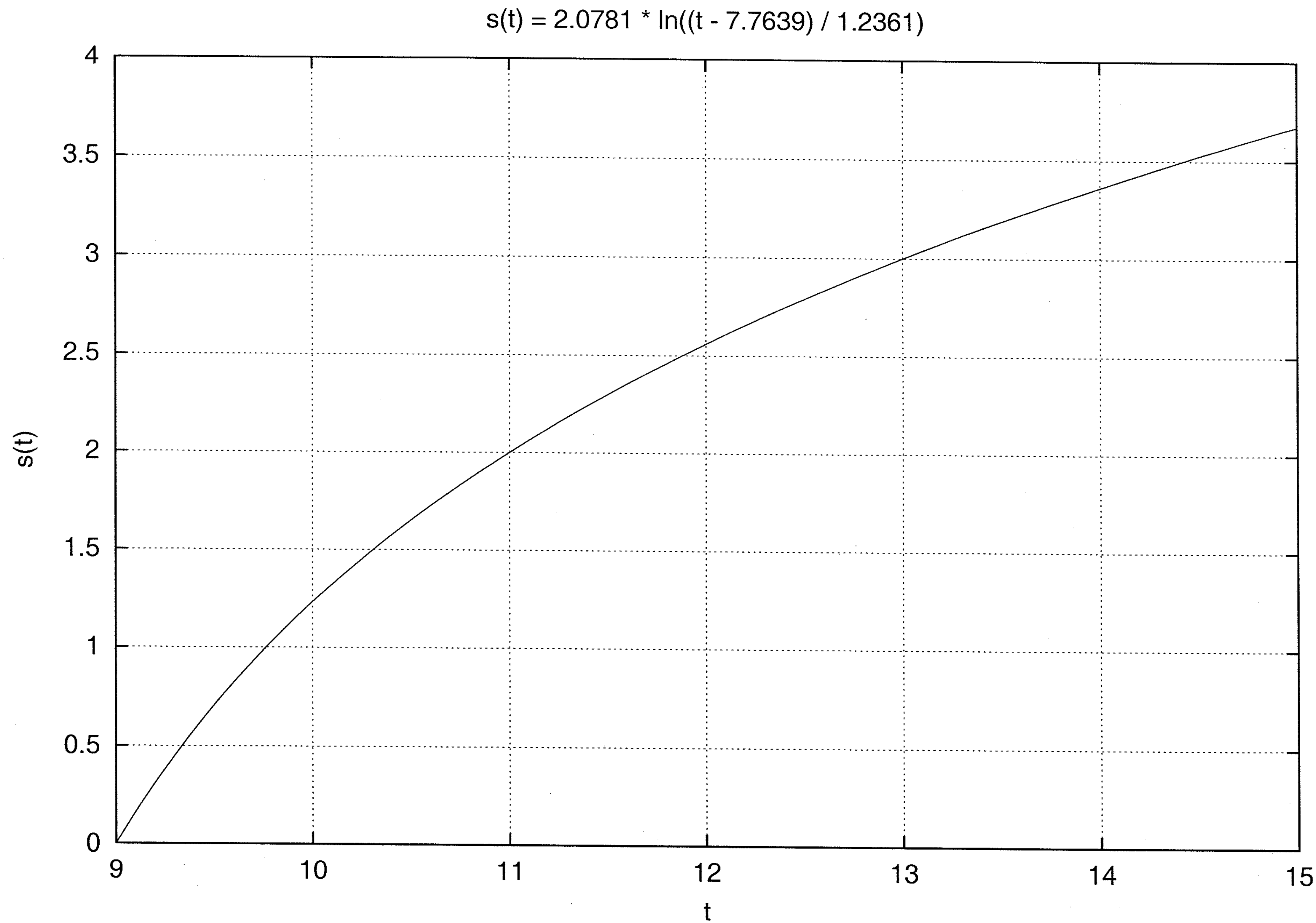
So $t_0 = \dfrac{20 \pm \sqrt{400 - 4(95)}}{2} = \dfrac{20 \pm 4.4721}{2}$

$t_0 = 7.7639$ hrs or 12.2361 hrs

only the first makes sense since the plow started at $t = 9$

$t_0 = 7.7639$ hrs or $\sim 7{:}46$ AM

Started snowing at $\underline{7{:}46 \text{ AM}}$


s(t) = 2.0781 * ln((t - 7.7639) / 1.2361)
s(t)
t
0
0.5
1
1.5
2
2.5
3
3.5
4
9
10
11
12
13
14
15