2.2 Separable Equations

We now consider first order DEs which, while they may not be linear, do have a property which often can be exploited.

The DE $\frac{dy}{dx} = f(x,y)$ can be rewritten

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

If it turns out that M is strictly a function of x and N is strictly a function of y

$$M(x) + N(y)\frac{dy}{dx} = 0$$

we say that the equation is <u>separable</u>. In differential form:

$$M(x)\,dx = -N(y)\,dy$$

If each side can be integrated and then if y can be isolated we have a solution.

<u>Ex</u> Solve $\frac{dy}{dx} = xy^2$ subject to $y(0) = -1$

1. Separate $\frac{dy}{y^2} = x\,dx$

2. Integrate $\int \frac{dy}{y^2} = \int x\,dx \rightarrow -\frac{1}{y} = \frac{x^2}{2} + C$

To find c, apply the initial condition

$$-\frac{1}{(-1)} = \frac{0}{2} + C \rightarrow C = 1$$

So

$$-\frac{1}{y} = \frac{x^2}{2} + 1 \longrightarrow \frac{1}{y} = -\frac{x^2}{2} - 1$$

So

$$y = \frac{1}{-\frac{x^2}{2} - 1} = \frac{-2}{x^2+2}$$

$$\boxed{y = \frac{-2}{x^2+2}}$$

Note that it may be difficult to solve for y explicitly after integrating.

Ex Solve $\frac{dy}{dx} = \frac{y-1}{x+3}$ subject to $y(-1) = 0$

$$\frac{dy}{y-1} = \frac{dx}{x+3}$$

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+3}$$

$$\ln|y-1| = \ln|x+3| + C$$

use initial condition

$$\ln|0-1| = \ln|-1+3| + C$$

$$0 = \ln 2 + C$$

$$-\ln 2 = C$$

so $$\ln|y-1| = \ln|x+3| - \ln 2 = \ln\left|\frac{x+3}{2}\right|$$

$$|y-1| = \left|\frac{x+3}{2}\right|$$

$$y - 1 = \pm \frac{x+3}{2}$$
$$y = 1 \pm \frac{x+3}{2} = \frac{2 \pm (x+3)}{2}$$

So $$y = \frac{x+5}{2} \quad \text{or} \quad y = \frac{-x-1}{2} = -\frac{x+1}{2}$$

but only the second satisfies the initial condition

$$\boxed{y = -\frac{x+1}{2}}$$

Note: if the constant of integration is retained until the last step this is the only answer that appears.

Ex Solve $y' = 2x\cos^2 y \qquad y(0) = \pi/4$

$$\frac{dy}{\cos^2 y} = 2x\,dx$$

$$\int \sec^2 y\,dy = \int 2x\,dx$$

$$\tan y = x^2 + C$$
$$\tan\left(\frac{\pi}{4}\right) = 0 + C \quad \to \quad C = 1$$

$$\tan y = x^2 + 1$$

$$\boxed{y = \tan^{-1}(x^2+1)}$$

$$\boxed{\text{Note that } \quad \frac{\pi}{4} \le y < \frac{\pi}{2}}$$

#22 $y' = \frac{3x^2}{3y^2-4}$ $\quad y(1)=0$

$$(3y^2-4)\,dy = 3x^2\,dx$$

integrating

$$y^3 - 4y = x^3 + C$$

$$0^3 - 4(0) = 1^3 + C \rightarrow C = -1$$

so

$$\boxed{y^3 - 4y = x^3 - 1}$$

To find interval in which the solution is valid, we determine where y' is undefined i.e. where $dx/dy = 0$

$$\frac{3y^2-4}{3x^2} = 0 \rightarrow 3y^2 - 4 = 0 \rightarrow y^2 = \frac{4}{3} \rightarrow y = \pm\frac{2}{\sqrt{3}}$$

So $\pm\left(\left[\frac{2}{\sqrt{3}}\right]^3 - 4\frac{2}{\sqrt{3}}\right) = x^3 - 1$

$$\pm\left(\frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}}\right) = \pm\frac{8 - 8\cdot 3}{3\sqrt{3}} = \mp\frac{16}{3\sqrt{3}} = x^3 - 1$$

or

$$|x^3 - 1| = \frac{16}{3\sqrt{3}}$$

gives endpoints of the domain

Since $x=1$ is in the domain, we need

$$|x^3 - 1| \le 16/3\sqrt{3}$$

$$-\frac{16}{3\sqrt{3}} < x^3 - 1 < \frac{16}{3\sqrt{3}}$$

$$1 - 16/3\sqrt{3} < x^3 < 1 + 16/3\sqrt{3}$$

$$-2.079 < x^3 < 4.079 \quad \text{(approx)}$$

$$\boxed{-1.276 < x < 1.598} \quad \text{(approx)}$$