# 2.1 Linear Equations



We begin by considering $1^{st}$ order linear equations.

$$y' = f(x,y) \qquad \text{general } 1^{st} \text{ order eq.}$$

$$y' + p(x)\,y = g(x) \qquad 1^{st} \text{ order linear eq.}$$

<u>Ex</u> Solve $y' + 2y = 1$

$$\frac{dy}{dx} = 1 - 2y$$

$$\frac{dy}{1-2y} = dx$$

$$\int \frac{dy}{1-2y} = \int dx$$

$u = 1-2y$

$du = -2dy$ $\qquad -\frac{1}{2}\int \frac{du}{u} = x + c$

$$\ln|u| = -2x + c$$

$$|u| = e^{-2x+c} = e^c e^{-2x} = Ce^{-2x}$$

$$u = \pm Ce^{-2x} = Ce^{-2x}$$

$$1 - 2y = Ce^{-2x}$$

$$-2y = Ce^{-2x} - 1$$

$$y = Ce^{-2x} + \tfrac{1}{2}$$

general solution

$$\boxed{y = ce^{-2x} + \tfrac{1}{2}}$$

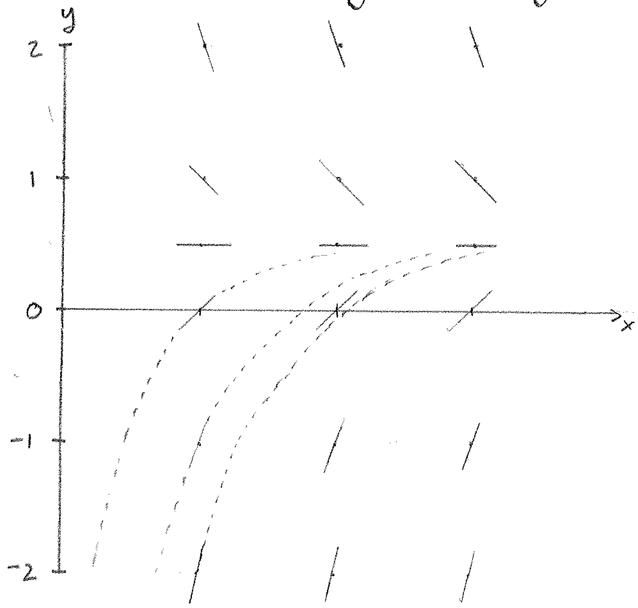
Check: $y' = -2ce^{-2x}$

$$y' + 2y = -2ce^{-2x} + 2(ce^{-2x} + \tfrac{1}{2}) = 1 \quad \checkmark$$

Notice that the solution has an arbitrary constant $c$ – There are actually a family of curves.

$$y' = 1 - 2y$$

DIRECTION FIELD




| $y$ | $y'$ |
|---|---|
| $-2$ | $5$ |
| $-1$ | $3$ |
| $0$ | $1$ |
| $\frac{1}{2}$ | $0$ |
| $1$ | $-1$ |
| $2$ | $-3$ |

All curves asymptotically approach $y = \frac{1}{2}$.

We can "pin down" our solution by requiring that the solution contain a particular point. For example if $\underbrace{y(1) = 0}$ then

initial condition

$$y = ce^{-2(1)} + \tfrac{1}{2} = 0$$
$$ce^{-2} = -\tfrac{1}{2}$$
$$c = -\tfrac{e^2}{2}$$

so

$$y = \left(-\frac{e^2}{2}\right) e^{-2x} + \frac{1}{2}$$

The general $1^{st}$ order linear eq is

$$y' + p(x)\,y = g(x) \qquad (*)$$

where $p(x)$ and $g(x)$ are arbitrary functions of $x$.

Recall the product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.
We now mult $(*)$ by a function $\mu(x)$ and try and determine $\mu$ so that we can work the product rule backwards

$$\mu(x)\,y' + \mu(x)\,p(x)\,y = \mu(x)\,g(x)$$

Want $\frac{d}{dx}(Sy) = Sy' + S'y = \mu y' + \mu p y$

so $S = \mu$

$S' = \mu p$

since the coefs of $y'$ must be eq on both sides of the equality. Same for $y$.

$$\mu' = \mu p$$

$$\frac{d\mu}{\mu} = p\,dx$$

$$\ln|\mu| = \int p\,dx + c$$

$$|\mu| = e^{\int p\,dx + c} = c\,e^{\int p\,dx}$$

$$\mu(x) = C\,e^{\int p(x)\,dx}$$

Since the value of $c$ is arbitrary, chose $c=1$ so that

$$\boxed{\mu(x) = e^{\int p(x)\,dx}}$$

<u>Integrating Factor</u>

Well, using this choice of $\mu(x)$, we find

$$\mu y' + \mu p y = (\mu y)' = \mu g$$

so $$\mu y = \int \mu g \, dx + c$$

$$\boxed{y = \frac{1}{\mu(x)}\left[\int \mu(x)\, g(x)\, dx + c\right]}$$

<u>Ex</u> Solve $y' + 2x y = e^{-x^2}$

1. find $\mu(x) = e^{\int 2x\,dx} = e^{x^2}$

2. $$e^{x^2}y' + 2x e^{x^2} y = e^{x^2}e^{-x^2}$$
$$(e^{x^2}y)' = 1$$

$$e^{x^2}y = \int dx$$
$$= x + c$$
$$\boxed{y = x e^{-x^2} + ce^{-x^2}}$$

Check $y' = -2xe^{-x^2}x + e^{-x^2} - 2xce^{-x^2}$

$$y' + 2xy = -2x^2e^{-x^2} - 2xce^{-x^2} + e^{-x^2} + 2x(xe^{-x^2} + ce^{-x^2})$$
$$= \cancel{-2x^2e^{-x^2}} \cancel{-2xce^{-x^2}} + e^{-x^2} \cancel{+2x^2e^{-x^2}} \cancel{+2xce^{-x^2}} = e^{-x^2} \checkmark$$

Of course, if an initial condition is supplied then we can determine the arbitrary constant in the solution.

E.g. $y(0) = 2$

$$2 = 0e^{-0^2} + ce^{-0^2} = c$$

$$2 = c$$

So $y = xe^{-x^2} + 2e^{-x^2} = (x+2)e^{-x^2}$