1.2 Solutions of Some Differential Equations

We "solve" a differential equation such as $dy/dt = f(t, y)$ by finding a function $y(t)$ that satisfies it.

Example: $dy/dt = 2y$

We want to rewrite this so all "y"s appear on one side of an equation and all "t"s appear on the other

$$\frac{dy}{dt} = 2y$$

$$\frac{1}{2y}\frac{dy}{dt} = 1$$

$$\frac{1}{2y}\,dy = dt$$

Now integrate

Hint: Find a Calculus book - we will be doing many different integrals.

$$\int \frac{1}{2y}\,dy = \int dt$$

$$\frac{1}{2}\int \frac{dy}{y} = t + C$$

$$\frac{1}{2}\ln|y| = t + c$$

Now solve for y

$$\ln|y| = 2t + 2c$$

$$|y| = e^{2t+2c}$$

$$= e^{2t}e^{2c}$$

What will we do with the absolute value signs? Note that e^{2c} is a positive constant. If we replace it with a new constant C that can be $\pm$ we have $\rightarrow$

$$y = Ce^{2t}$$

This is a solution — actually it represents a family of solutions of $dy/dt = 2y$ because C can be any constant (even zero)

Check: $y' = 2Ce^{2t} = 2y$ ✓

Suppose, along with the differential equation we also have an initial condition $y(0) = 1$. This will uniquelly determine the value of C and hence the function y.

$$y(0) = 1 = Ce^{2(0)} = Ce^0 = C$$

so $C = 1$.

The pair $\begin{cases} dy/dt = 2y \\ y(0) = 1 \end{cases}$ is called an initial value problem

or IVP.

Ex $\quad dy/dx = 3y - 2$

$$\frac{1}{3y-2}\,dy = dx$$

$$\int \frac{1}{3y-2}\,dy = \int dx$$

$$\frac{1}{3}\ln|3y-2| = x + C$$

(chain rule)

$$\frac{1}{3}\ln|3y-2| = x + c$$

$$\ln|3y-2| = 3x + 3c$$

$$|3y-2| = e^{3x}e^{3c}$$

$$3y-2 = Ce^{3x}$$

$$3y = Ce^{3x} + 2$$

$$y = \frac{C}{3}e^{3x} + 2/3$$

or, since $C/3$ is yet another constant,

$$\boxed{y = ce^{3x} + 2/3}$$

Check $y' = 3ce^{3x}$

same

$$3y-2 = 3\left[ce^{3x} + 2/3\right] - 2$$
$$= 3ce^{3x} + 2 - 2$$
$$= 3ce^{3x}$$

if $y(0) = 2$

$$2 = Ce^{0} + 2/3$$
$$2 = C + 2/3$$
$$C = 4/3$$

$$y = \frac{4}{3}e^{3x} + \frac{2}{3}$$