N Roots of a Complex Number

Motivating Example: Solve for z: $z^4 = 1$

Aside: Solve $z^n - a = 0$ or $z^n = a \rightarrow z = \sqrt[n]{a}$

use polar form $z = \rho[\cos\alpha + i\sin\alpha] = \rho e^{i\alpha}$

let $a = re^{i\theta}$: if a is real then $\theta = 0$ and $a = r$

$z^n = a \rightarrow \rho^n e^{in\alpha} = re^{i\theta}$

magnitude part

Note that $|e^{ix}| = 1$ for all values of x : $\rho \underbrace{e^{i\alpha}}$

direction part

So $\rho^n = r$, $e^{in\alpha} = e^{i\theta}$

$\rho = \sqrt[n]{r}$ (unique, real, n^{th} root of r)

$n\alpha = \theta \pm k(2\pi)$ since any multiple of 2π just brings us around again

$\alpha = \frac{\theta}{n} \pm k\frac{2\pi}{n}$

Notice that $e^{i\alpha}$ will have n different values, corresponding to $k = 0, 1, \cdots, n-1$

$\alpha = \frac{\theta}{n}, \frac{\theta}{n} + \frac{2\pi}{n}, \frac{\theta}{n} + \frac{4\pi}{n}, \frac{\theta}{n} + \frac{6\pi}{n}, \cdots, \frac{\theta}{n} + \frac{(2n-2)\pi}{n}$

So $z = \sqrt[n]{r}[\cos\alpha + i\sin\alpha]$ for $\alpha = \frac{\theta}{n} + k\frac{2\pi}{n}$ $k = 0, \cdots, n-1$

N Roots of a Complex Number

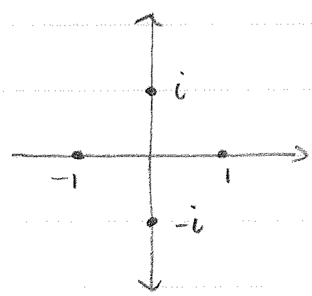
<u>Ex</u> $\quad z^4 - 1 = 0 \quad \Rightarrow \quad z^4 = 1$

$$z = \rho e^{i\alpha} \qquad 1 = a e^{i\theta} = 1 e^{i\cdot 0}$$

$$z^4 = \rho^4 e^{i4\alpha} = 1 e^{i0} \quad \Rightarrow \quad \rho^4 = 1, \quad 4\alpha = 0 \pm k(2\pi)$$

$$\rho = 1 \qquad \alpha = k\frac{2\pi}{4} \qquad k = 0, 1, 2, 3$$

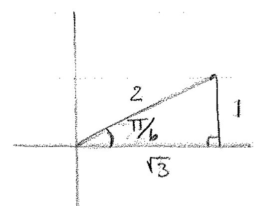
$$\alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$



$z = 1e^{i\alpha}, \quad \alpha = 0, \pi/2, \pi, 3\pi/2$

$z = 1, i, -1, -i$

<u>Ex</u> $\quad z^3 - (\sqrt{3} + i) = 0 \quad \Rightarrow \quad z^3 = \sqrt{3} + i$



$$z = \rho e^{i\alpha} \qquad \sqrt{3} + i = 2e^{i\frac{\pi}{6}}$$

$$z^3 = \rho^3 e^{i3\alpha}$$

$$\rho^3 = 2 \rightarrow \rho = \sqrt[3]{2}$$

$$3\alpha = \frac{\pi}{6} \pm k(2\pi) \longrightarrow \alpha = \frac{\pi}{18} \pm k\frac{2\pi}{3} \qquad k = 0, 1, 2$$

$$\alpha = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}$$

$$z = \sqrt[3]{2}\left[\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right],$$

$$\sqrt[3]{2}\left[\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right]$$

$$\sqrt[3]{2}\left[\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18}\right]$$