A SHORT JUSTIFICATION OF SEPARATION OF VARIABLES

A first order differential equation is *separable* if it can be written in the form

$$m(x) + n(y)\frac{dy}{dx} = 0. \tag{1}$$

The standard approach to solve this equation for $y(x)$ is to treat dy/dx as a fraction and move all quantities involving x to the right side and all quantities involving y to the left. Each side is then integrated with respect to the appropriate variable, giving

$$\int n(y)dy = -\int m(x)dx. \tag{2}$$

The resulting equation can (hopefully) be solved for y.

There are two surmountable difficulties with this approach. First, we need to justify that dy/dx, which represents the rate of change of y with respect to x, can be treated as a fraction and split into two *differentials* dy and dx. Second, are we really justified in integrating each side of the equation with respect to different variables? To see that this approach is valid we'll show that (2) can be obtained without either of these problematic operations.

Rearranging (1) and integrating both sides with respect to x we have

$$\int n(y)\frac{dy}{dx}dx = -\int m(x)dx \tag{3}$$

Clearly the right-hand-side of this equation matches the right-hand-side of (2). Now suppose $N(y)$ is an antiderivative with respect to y of $n(y)$; that is, $N'(y) = n(y)$. Then, by the chain rule

$$\frac{d}{dx}N(y) = n(y)\frac{dy}{dx}.$$

Swapping sides, integrating with respect to x, and noting that $N(y) = \int n(y)dy$ we have

$$\int n(y)\frac{dy}{dx}dx = N(y) = \int n(y)dy$$

so we see that equations (2) and (3) are equivalent. This justifies the standard approach of using the differential form given in equation (2) to solve separable equations.

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