

Integration by Substitution

Find the following derivatives

$$\frac{d}{dx} e^{x^2} = e^{x^2} \cdot (2x) = 2x e^{x^2}$$

$$\frac{d}{dx} \cos(2x+1) = -\sin(2x+1) \cdot (2) = -2 \sin(2x+1)$$

$$\frac{d}{dx} \ln(x^2+4) = \frac{1}{x^2+4} \cdot (2x) = \frac{2x}{x^2+4}$$

We used the chain rule to evaluate each of these derivatives.

Consider $\int 2x e^{x^2} dx$

From the work above we know that this evaluates to $e^{x^2} + C$.
How could we do this without having just done the corresponding derivative?

Looking at the integrand we observe that $2x$ is the derivative of x^2 .

We can try the substitution (change of variable) $u(x) = x^2$

$$u = x^2$$

$$du/dx = 2x$$

$$\begin{aligned} \text{So } \int 2x e^{x^2} dx &= \int \frac{du}{dx} e^u dx = \int e^u du = e^u + C \\ &= \underline{e^{x^2} + C} \end{aligned}$$

$$\text{Ex } \int -2 \sin(2x+1) dx \quad \text{Let } u = 2x+1 \text{ so } du = 2 dx$$

$$= \int -\sin u du = \cos u + C = \underline{\cos(2x+1) + C}$$

$$\text{Ex } \int \frac{2x}{x^2+4} dx \quad \text{Let } u = x^2+4 \text{ so } du = 2x dx$$

$$= \int \frac{1}{u} du = \ln|u| + C = \ln|x^2+4| + C = \underline{\ln(x^2+4) + C}$$

Integration by Substitution

2

Ex $\int x \sqrt{3x^2+1} dx$

Here we try $u=3x^2+1$ so $du=6x dx$

We don't have $6x$ in the integrand, but we can get it if we multiply by 6 inside the integral and divide by 6 outside.

$$\begin{aligned}\int x \sqrt{3x^2+1} dx &= \frac{1}{6} \int 6x \sqrt{3x^2+1} dx = \frac{1}{6} \int \sqrt{u} du \\ &= \frac{1}{6} \int u^{1/2} du = \frac{1}{6} \cdot \frac{2}{3} u^{3/2} + C \\ &= \frac{1}{9} u^{3/2} + C = \underline{\underline{\frac{1}{9} (3x^2+1)^{3/2} + C}}\end{aligned}$$

Check $\frac{d}{dx} \frac{1}{9} (3x^2+1)^{3/2} + C = \frac{1}{9} \cdot \frac{3}{2} (3x^2+1)^{1/2} (6x)$
 $= \frac{1}{6} \sqrt{3x^2+1} (6x)$
 $= x \sqrt{3x^2+1}$

This substitution worked because we let u equal a quadratic function of x and there was a corresponding linear function of x in the integrand.

Principle: When choosing u , the derivative of u (up to a scalar multiple) must appear as a factor in the integrand.

Ex $\int \frac{x}{x^2+1} dx$ let $u=x^2+1$ so $du=2x dx$

$$\begin{aligned}&= \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C \\ &= \underline{\underline{\frac{1}{2} \ln|x^2+1| + C}}\end{aligned}$$

Integration by Substitution

3

Ex $\int \frac{1}{x^2+1} dx = \underline{\arctan x + C}$ Substitution is not used.

Ex $\int \frac{x^2}{x^2+1} dx$ $u = x^2+1 \rightarrow du = 2x dx$
Cannot use this approach since we have x^2 in numerator rather than x .

We can divide
$$\begin{array}{r} 1 \\ x^2+1 \overline{) x^2} \\ \underline{-(x^2+1)} \\ -1 \end{array}$$
 so

$$= \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \underline{x - \arctan x + C}$$

Ex $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ $u = \sqrt{x}$ $du = \frac{1}{2} x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx$

$$= 2 \int \frac{\cos \sqrt{x}}{2\sqrt{x}} dx = 2 \int \cos u du$$
$$= 2 \sin u + C = \underline{2 \sin \sqrt{x} + C}$$

Ex $\int x\sqrt{x-1} dx$ This one does not quite fit the rules shown so far — but we can use some algebraic manipulation

Let $u = x-1$. This means $du = dx$

What about the factor of x ? If $u = x-1$ then $x = u+1$.

$$\begin{aligned} \int x\sqrt{x-1} dx &= \int (u+1)\sqrt{u} du = \int u^{3/2} + u^{1/2} du \\ &= \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C \\ &= \underline{\underline{\frac{2}{5} (x-1)^{5/2} + \frac{2}{3} (x-1)^{3/2} + C}} \end{aligned}$$

Integration by Substitution

4

We can use substitution with definite integrals as well

$$\text{Ex } \int_1^3 x \sqrt{x^2-1} \, dx \quad u = x^2-1, \, du = 2x \, dx$$

We need to change the limits so that they correspond to the change in variable

$$x=1 \rightarrow u = 1^2-1 = 0$$

$$x=3 \rightarrow u = 3^2-1 = 8$$

$$\begin{aligned} \text{So } \frac{1}{2} \int_1^3 2x \sqrt{x^2-1} \, dx &= \frac{1}{2} \int_0^8 \sqrt{u} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_0^8 = \frac{1}{3} u^{3/2} \Big|_0^8 \\ &= \frac{1}{3} [8^{3/2} - 0^{3/2}] = \frac{1}{3} (\sqrt{8})^3 = \underline{\underline{\frac{8}{3} (\sqrt{2})^3}} \end{aligned}$$

$$\text{Ex } \int_0^{\pi/3} \tan x \, dx = \int_0^{\pi/3} \frac{\sin x}{\cos x} \, dx$$

$$u = \cos x, \, du = -\sin x \, dx$$

$$u(0) = 1, \, u(\pi/3) = 1/2$$

$$\begin{aligned} &= -\int_1^{1/2} \frac{1}{u} \, du = \int_{1/2}^1 \frac{1}{u} \, du = \ln|u| \Big|_{1/2}^1 = \ln 1 - \ln(1/2) \\ &= 0 - (-\ln 2) = \underline{\underline{\ln 2}} \end{aligned}$$