Trigonometric Identities

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1 Trigonometric Identities you must remember

The "big three" trigonometric identities are

$$\sin^2\theta + \cos^2\theta = 1 \tag{1}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{2}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3}$$

Using these we can derive many other identities. Even if we commit the other useful identities to memory, these three will help be sure that our signs are correct, etc.

2 Two more easy identities

From equation (1) we can generate two more identities. First, divide each term in (1) by $\cos^2\theta$ (assuming it is not zero) to obtain

$$\tan^2\theta + 1 = \sec^2\theta. \tag{4}$$

When we divide by $\sin^2\theta$ (again assuming it is not zero) we get

$$1 + \cot^2\theta = \csc^2\theta. \tag{5}$$

3 Identities involving the difference of two angles

From equations (2) and (3) we can get several useful identities. First, recall that

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta. \tag{6}$$

From (2) we see that

$$\begin{aligned} \sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) \\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \end{aligned}$$

which, using the relationships in (6), reduces to

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{7}$$

In a similar way, we can use equation (3) to find

$$\begin{aligned} \cos(\alpha-\beta) &= \cos(\alpha+(-\beta)) \\ &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \end{aligned}$$

which simplifies to

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \tag{8}$$

Notice that by remembering the identities (2) and (3) you can easily work out the signs in these last two identities.

4 Identities involving products of Sines and Cosines

If we now add equation (2) to equation (7)

$$\begin{aligned} \sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\ +(\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta) \end{aligned}$$

we find

$$\sin(\alpha-\beta)+\sin(\alpha+\beta)=2\sin\alpha\cos\beta$$

and dividing both sides by 2 we obtain the identity

$$\sin\alpha\cos\beta=\frac{1}{2}\sin(\alpha-\beta)+\frac{1}{2}\sin(\alpha+\beta). \tag{9}$$

In the same way we can add equations (3) and (8)

$$\begin{aligned} \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \\ +(\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta) \end{aligned}$$

to get

$$\cos(\alpha-\beta)+\cos(\alpha+\beta)=2\cos\alpha\cos\beta$$

which can be rearranged to yield the identity

$$\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta). \tag{10}$$

Suppose we wanted an identity involving $\sin\alpha\sin\beta$. We can find one by slightly modifying the last thing we did. Rather than adding equations (3) and (8), all we need to do is subtract equation (3) from equation (8):

$$\begin{aligned} \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \\ -(\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta) \end{aligned}$$

This gives

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$$

or, in the form we prefer,

$$\sin\alpha\sin\beta=\frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta). \tag{11}$$

5 Double angle identities

Now a couple of easy ones. If we let $\alpha=\beta$ in equations (2) and (3) we get the two identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \tag{12}$$
$$\cos 2\alpha = \cos^2\alpha-\sin^2\alpha. \tag{13}$$

6 Identities for Sine squared and Cosine squared

If we have $\alpha = \beta$ in equation (10) then we find

$$\begin{aligned}\cos\alpha\cos\alpha &= \frac{1}{2}\cos(\alpha-\alpha)+\frac{1}{2}\cos(\alpha+\alpha)\\ \cos^2\alpha &= \frac{1}{2}\cos 0+\frac{1}{2}\cos 2\alpha.\end{aligned}$$

Simplifying this and doing the same with equation (11) we find the two identities

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha), \tag{14}$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha). \tag{15}$$

7 Identities involving tangent

Finally, from equations (2) and (3) we can obtain an identity for $\tan(\alpha+\beta)$:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$$

Now divide numerator and denominator by $\cos\alpha\cos\beta$ to obtain the identity we wanted:

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}. \tag{16}$$

We can get the identity for $\tan(\alpha-\beta)$ by replacing β in (16) by $-\beta$ and noting that tangent is an odd function:

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}. \tag{17}$$

8 Summary

There are many other identities that can be generated this way. In fact, the derivations above are not unique — many trigonometric identities can be obtained many different ways. The idea here is to be very familiar with a small number of identities so that you are comfortable manipulating and combining them to obtain whatever identity you need to.

9 Table of Identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

4. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
5. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
6. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
7. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

8. $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta)$
9. $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$
10. $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$

11. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
12. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

13. $\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$
14. $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$

15. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
16. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$